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Shifted Tableaux and the Projective Representations of Symmetric Groups

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INTRODUCTION

The projective representations of the symmetric and alternating groups were originally studied by Schur. In his fundamental paper [S], Schur derived degree and character formulas for these representations remarkably similar in style to the corresponding formulas for ordinary representations due to Frobenius. Subsequently, the ordinary representations of S_n have become part of a rich theory connecting symmetric functions, representations, and the combinatorics of tableaux, but the projective representations have not fared as well.

Recently, a combinatorial theory of shifted tableaux parallel to the theory of ordinary tableaux has been developed independently by Sagan [Sa1] and Worley [Wo]. This theory includes shifted versions of the Robinson–Schensted–Knuth correspondence, Greene's invariants, Knuth relations, and Schützenberger's *jeu de taquin*. With a few exceptions, the connections between this theory and the projective representations of S_n have not been well understood, at least in comparison with the corresponding theories for ordinary tableaux and linear representations.

In this paper, we present a reconciliation of the projective and combinatorial theories that more deeply explains the connections. Among the consequences of this reconciliation are a new proof of Schur's description of the irreducible characters, a projective analogue of induction from parabolic subgroups, a shifted analogue of the Littlewood–Richardson

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rule, and a projective generalization of Littlewood's "inner product" of symmetric functions.

A more detailed summary follows.

The first three sections are mostly expository. We first give a (modernized) review of Schur's theory of projective representations of finite groups, and construct the representation groups of S_n. These are the minimal central extensions $\tilde{S}_n$ (double covers, if $n \geqslant 4$) whose ordinary representations are equivalent to the projective representations of S_n. In Section 2, we describe the conjugacy class of $\tilde{S}_n$, as well as those of $\tilde{A}_n$ (the subgroup that doubly covers the alternating group) and $\tilde{S}_n^J$ (the double covers of parabolic subgroups). Since the description of these conjugacy classes is so fundamental in what follows, and since it is difficult to extract this information from [S], we have included complete proofs. In Section 3, we construct the basic spin representations via homomorphisms of the form

$$\tilde{S}_n \longrightarrow \mathscr{C}_{n-1}^* \xrightarrow{\rho} GL(V),$$

where ρ denotes any irreducible representation of (the multiplicative group of) the Clifford algebra $\mathscr{C}_{n-1}$. Although this construction is implicit in Schur's development (cf. [S, p. 199]), the derivation we give takes full advantage of the simplicity afforded by Clifford algebras. An alternative approach has been given by Morris [Mo1].

We next consider the problem of inducing representations from $\tilde{S}_n^J$ to $\tilde{S}_n$. In the case of ordinary representations, the parabolic (or Young) subgroups S_n^J are direct products of smaller symmetric groups. Hence, given an S_k-module V and an S_{n-k}-module W, the outer tensor product $V \otimes W$ is a module for a parabolic subgroup ($\cong S_k \times S_{n-k}$) and thus may be easily induced up to S_n. This technique is valuable for constructing irreducible S_n-modules, since the λth irreducible representation of S_n occurs without multiplicity as the "leading term" of the induced module $(1_{\lambda_1} \otimes \cdots \otimes 1_{\lambda_l}) \uparrow S_n$, where 1_k denotes the trivial S_k-module. Furthermore, the λth irreducible representation may be isolated from this induced module via Young's symmetrizers.

In Section 4, we introduce a projective analogue of the outer tensor product we call the reduced Clifford product. Given (irreducible) modules $V_1, ..., V_l$ for $\tilde{S}_{\lambda_1}, ..., \tilde{S}_{\lambda_l}$, we show that the reduced Clifford product of $V_1, ..., V_l$ is an (irreducible) $\tilde{S}_n^J$-module, where $S_n^J \cong S_{\lambda_1} \times \cdots \times S_{\lambda_l}$. (Since $\tilde{S}_n^J \not\cong \tilde{S}_{\lambda_1} \times \cdots \times \tilde{S}_{\lambda_l}$, the outer tensor product is entirely inadequate for this purpose.) Conversely, we show that every irreducible $\tilde{S}_n^J$-module is a reduced Clifford product. In Theorem 7.2 (an analogue of Young's rule), we will see that the λth irreducible projective representation of S_n occurs without multiplicity as the "leading term" of the induced module $R^\lambda \uparrow \tilde{S}_n$, where R^λ denotes the reduced Clifford product of the basic spin representations of $\tilde{S}_{\lambda_1}, ..., \tilde{S}_{\lambda_l}$. Whether there exists a projective analogue of Young's symmetrizers that could isolate the λth irreducible representation from $R^\lambda \uparrow \tilde{S}_n$ remains an intriguing open problem. In this context, we should remark that a construction of the irreducible projective S_n-modules has been very recently announced by Nazarov [N].

In the remainder of the paper, the focus shifts to characters, symmetric functions, and tableaux. We show (Section 5) that the connection between projective characters and symmetric functions can be understood in terms of a linear map

$$Z'_n \xrightarrow{\text{ch}'} \Omega^n$$

between the space Z'_n of class functions spanned by projective characters, and a certain graded subalgebra $\Omega = \bigoplus_{n \geqslant 0} \Omega^n$ of the full algebra of symmetric functions. We show that the map ch′ satisfies many of the same properties as Frobenius' characteristic map for ordinary characters. In particular, it is (approximately) metric-preserving, and multiplication in Ω corresponds (approximately) to the reduced Clifford product of characters.

In Section 6, the combinatorial theory of Sagan and Worley is used to define Schur's symmetric functions Q_λ in terms of shifted tableaux. It is far from obvious that this definition agrees with Schur's [S, p. 225], or equivalently, with the $t = -1$ specialization of the Hall–Littlewood functions [L1]. The fact that these are equivalent follows from Macdonald's tableau description of $Q_\lambda(x; t)$ [Ma, III(5.11)]; however, use of this equivalence is unnecessary unless the reader wishes to independently confirm that our description of the irreducible projective characters agrees with Schur's. To emphasize the fact that our development does not need this equivalence, we have included proofs of a number of symmetric function identities that could otherwise have simply been quoted as $t = -1$ specializations of identities in [Ma].

We derive Schur's description of the irreducible characters in Section 7, along with the previously mentioned analogue of Young's rule. It is interesting to note that the irreducible characters of $\tilde{A}_n$ are almost unavoidably obtained as a corollary. We should also remark that Morris [Mo2] has derived a combinatorial recurrence, related to the Murnaghan–Nakayama rule, for computing the irreducible projective characters of S_n (see also [Mo3, H]). It is possible to give a proof of this recurrence using shifted tableaux, rather than the machinery of Hall–Littlewood functions used by Morris, but we have not included the details here. Character tables for $n \leqslant 13$ appear in [Mo1].

In Section 8, we apply Sagan and Worley's theory of shifted tableaux to prove an analogue of the Littlewood–Richardson rule that describes the Q-function expansion of the product $Q_\lambda Q_\mu$. From the theory we have developed, it will follow that this rule also describes the irreducible expansion of the induced character corresponding to the reduced Clifford product of the λth and μth projective representations.

Finally, we briefly consider (Section 9) the inner tensor product of $\tilde{S}_n$-modules, and define a related symmetric function operation that generalizes Littlewood's "inner product" [L2]. As an application, we derive a combinatorial description of the irreducible character expansion of the tensor product of the basic spin representation of $\tilde{S}_n$ with any irreducible S_n-module.

Conventions. All representations, vector spaces, and algebras will use the complex field. Exceptions to this rule will sometimes occur in discussing spaces of symmetric functions, but these exceptions will be of characteristic 0 and explicitly noted. All algebras will be associative and include an identity. We will use the notation A^* to indicate the multiplicative group of units of the algebra A.

1. The Representation Groups of S_n

A *projective representation* of a group G (assumed finite) is a homomorphism of G into the projective general linear group, $PGL(V) = GL(V)/\text{scalars}$. Equivalently, such a representation may be viewed as a map $P: G \to GL(V)$ such that

$$P(x)\,P(y) = c_{x,y} P(xy) \qquad (x, y \in G)$$

for suitable scalars $c_{x,y} \in \mathbf{C}^*$. The map $(x, y) \mapsto c_{x,y}$ is called a *factor set.* Note that associativity of $GL(V)$ implies

$$c_{x,y} c_{xy,z} = c_{x,yz} c_{y,z} \qquad (x, y, z \in G) \tag{1.1}$$

for any factor set c.

Two projective representations $P: G \to GL(V)$ and $Q: G \to GL(W)$ are equivalent if there is an invertible $S \in \mathrm{Hom}_{\mathbf{C}}(V, W)$ and a map $b: G \to \mathbf{C}^*$ such that

$$b_x SP(x)\, S^{-1} = Q(x)$$

for all $x \in G$. Equivalent projective representations are said to have equivalent factor sets. Thus, c, c': $G \times G \to \mathbf{C}^*$ are equivalent if they differ only by a factor $b_x b_y / b_{xy}$ for some $b: G \to \mathbf{C}^*$. The *Schur multiplier* of G is the abelian group of factor sets modulo equivalence; it is isomorphic to the second cohomology group $H^2(G, \mathbf{C}^*)$ (see [CR, Sect. 8] for more details).

For each factor set c there is an associated *twisted group algebra* $\mathbf{C}G^c$, with a basis $\{\alpha_x : x \in G\}$ indexed by G and multiplication defined by $\alpha_x\alpha_y = c_{x,y}\alpha_{xy}$. Associativity of $\mathbf{C}G^c$ is implied by (1.1). Note that a projective repesentation of G with factor set c is a $\mathbf{C}G^c$-module, and conversely. Since equivalent factor sets produce isomorphic algebras, we may assume without loss of generality that c is always chosen so that the distinguished basis $\{\alpha_x\}$ includes the identity of $\mathbf{C}G^c$.

It is not hard to show that twisted group algebras are semisimple (consider the trace of the action of $\mathbf{C}G^c$ on itself), so much of the same algebraic theory that applies to group algebras may be applied to projective representations. However, there is an alternative point of view, originally developed by Schur, which shows that it is possible to reduce the study of projective representations of G to the study of ordinary (i.e., linear) representations of certain covering groups of G.

Recall that a central extension of G is a pair (E, ψ) in which $\psi: E \to G$ is a surjective group homomorphism with $\ker \psi \subseteq ZE$, where ZE denotes the center of E. According to Schur's theory, every finite group G has central extensions (E, ψ) with the property that every projective representation P of G lifts to an ordinary representation $\hat{P}$ of E so that the diagram

$$\begin{tikzcd} E \arrow[r, "\hat{P}"] \arrow[d, "\psi"'] & GL(V) \arrow[d] \\ G \arrow[r, "P"] & PGL(V) \end{tikzcd}$$

commutes.

There exist such extensions with $\ker \psi \cong H^2(G, \mathbf{C}^*)$, and moreover, $H^2(G, \mathbf{C}^*)$ is the unique minimal possibility for $\ker \psi$ [CR, Sect. 11E]. These minimal central extensions are called *representation groups* of G.

To determine the representation groups of S_n, we first determine the associated twisted group algebras. Of course, $\mathbf{C}S_n$ is itself one of these algebras. We note that since S_n has a Coxeter group presentation

$$s_j^2 = 1; \qquad (s_j s_k)^2 = 1 \qquad |j-k| \geqslant 2; \qquad (s_j s_{j+1})^3 = 1, \tag{1.2}$$

where $s_1, ..., s_{n-1}$ denote the adjacent transpositions, it follows that these relations also define an algebra presentation of $\mathbf{C}S_n$.

We claim that there is another twisted group algebra, which we will denote by $\mathbf{C}S'_n$, with generators $\alpha_1, ..., \alpha_{n-1}$ corresponding to $s_1, ..., s_{n-1}$, and an algebra presentation

$$\alpha_j^2 = 1; \qquad (\alpha_j\alpha_k)^2 = -1 \qquad |j-k| \geqslant 2; \qquad (\alpha_j\alpha_{j+1})^3 = 1. \tag{1.3}$$

We postpone proving the existence of a corresponding factor set for this algebra until Section 3. Note that if $n \leqslant 3$, there are no relations of the form $(\alpha_j\alpha_k)^2 = -1$, so the algebras $\mathbf{C}S_n$ and $\mathbf{C}S'_n$ coincide. Otherwise, we claim

LEMMA 1.1. *The algebras $\mathbf{C}S_n$ and $\mathbf{C}S'_n$ are the only twisted group algebras for S_n. In particular, the Schur multiplier $H^2(S_n, \mathbf{C}^*)$ is of order* 2 ($n \geqslant 4$).

Proof. Let c be an S_n factor set, and let $\alpha_j \in \mathbf{C}S_n^c$ denote the element corresponding to $s_j \in S_n$ ($1 \leqslant j < n$). Since $\alpha_j^2 \in \mathbf{C}^*$, we may assume $\alpha_j^2 = 1$ after suitably rescaling the factor set. We claim that such a scaling must satisfy

$$(\alpha_j \alpha_k)^2 = \pm 1 \qquad (|j-k| \geqslant 2); \qquad (\alpha_j \alpha_{j+1})^3 = \pm 1;$$

i.e., the factor set is ± 1-valued. To see this, observe that $s_j s_k s_j = s_k$ implies $\alpha_j \alpha_k \alpha_j = c\alpha_k$ for some $c \in \mathbf{C}^*$. Hence, $1 = (\alpha_j \alpha_k \alpha_j)^2 = (c\alpha_k)^2 = c^2$, so $(\alpha_j \alpha_k)^2 = c = \pm 1$. Similarly, we have $s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}$, so $\alpha_j \alpha_{j+1} \alpha_j = c\alpha_{j+1} \alpha_j \alpha_{j+1}$ for some $c \in \mathbf{C}^*$. Therefore, $1 = (\alpha_j \alpha_{j+1} \alpha_j)^2 = (c\alpha_{j+1} \alpha_j \alpha_{j+1})^2 = c^2$, so $(\alpha_j \alpha_{j+1})^3 = c = \pm 1$, as claimed.

Let s, $t \in S_n$ be a pair of conjugate permutations, and let σ, τ denote their counterparts in $\mathbf{C}S_n^c$. Since c is ± 1-valued, there is an invertible $\alpha \in \mathbf{C}S_n^c$ such that $\alpha\sigma\alpha^{-1} = \pm\tau$. If s and t are of even order, say $2m$, it follows that $\sigma^{2m} = \tau^{2m}$ ($= \pm 1$). In particular, since all of the elements $s_j s_k$ ($|j-k| \geqslant 2$) are conjugate involutions, it follows that $(\alpha_j \alpha_k)^2$ is a constant. Finally, note that by further rescaling the factor set (substituting $\alpha_j \to \pm\alpha_j$, if necessary), we may assume $(\alpha_j \alpha_{j+1})^3 = 1$. Thus, we have shown $\mathbf{C}S_n^c = \mathbf{C}S_n$ or $\mathbf{C}S'_n$, modulo equivalence of factor sets.

In case $n \geqslant 4$, the factor sets which produce $\mathbf{C}S_n$ and $\mathbf{C}S'_n$ cannot be equivalent. If $\alpha_1^2 = \alpha_3^2 = 1$, $(\alpha_1 \alpha_3)^2 = -1$, there is no way to rescale α_1 and α_3 so that $(\alpha_1 \alpha_3)^2 = +1$ without contradicting either $\alpha_1^2 = 1$ or $\alpha_3^2 = 1$. ∎

Define $\sigma_j = i\alpha_j$ ($i = \sqrt{-1}$) for $1 \leqslant j < n$, and observe that (1.3) implies

$$\sigma_j^2 = -1; \qquad (\sigma_j \sigma_k)^2 = -1 \qquad |j-k| \geqslant 2; \qquad (\sigma_{j+1})^3 = -1. \tag{1.4}$$

Given that (1.3) does define an algebra of dimension $n!$ (to be proved in Section 3), it follows that (1.4) determines a group $\tilde{S}_n$ of order $2 \cdot n!$ generated by $-1, \sigma_1, ..., \sigma_{n-1}$. Similarly, if we define $\sigma'_j = \alpha_j$ ($1 \leqslant j < n$), we obtain a group $\tilde{S}'_n$ of order $2 \cdot n!$ generated by $-1, \sigma'_1, ..., \sigma'_{n-1}$ subject to the relations

$$(\sigma'_j)^2 = 1; \qquad (\sigma'_j \sigma'_k)^2 = -1 \qquad |j-k| \geqslant 2; \qquad (\sigma'_j \sigma'_{j+1})^3 = 1. \tag{1.5}$$

Observe that a $\mathbf{C}S'_n$-module V is equivalent to an ordinary representation of $\tilde{S}_n$ (or $\tilde{S}'_n$) with -1 represented faithfully in $GL(V)$. Such modules will be called *spin representations*. Similarly, a $\mathbf{C}S_n$-module V (i.e., an ordinary representation of S_n) is equivalent to a representation of $\tilde{S}_n$ (or $\tilde{S}'_n$) with -1 represented by $1 \in GL(V)$. We remark that since -1 is central in $\tilde{S}_n$ and $\tilde{S}'_n$, Schur's lemma implies that every irreducible representation of $\tilde{S}_n$ or $\tilde{S}'_n$ must be one of these two types. Since Lemma 1.1 shows that every projective representation of S_n is either a $\mathbf{C}S_n$- or $\mathbf{C}S'_n$-module, we have proved that $\tilde{S}_n$ and $\tilde{S}'_n$ are representation groups of S_n (assuming $n \geqslant 4$).

THEOREM 1.2 (Schur [S, p. 166]). *The only representation groups of* S_n $(n \geqslant 4)$ *are* $\tilde{S}_n$ *and* $\tilde{S}'_n$.

Proof. Let (E, ψ) be a central extension of S_n. If E is a representation group, then $|\ker \psi| = 2$, since Lemma 1.1 shows that there are only two twisted group algebras. Write $\ker \psi = \{\pm 1\}$, and let $e_j \in E$ be an element covering $s_j \in S_n$, i.e., $\psi e_j = s_j$. In view of the Coxeter presentation (1.2), we have

$$e_j^2 = \pm 1; \qquad (e_j e_k)^2 = \pm 1 \ |j-k| \geqslant 2; \qquad (e_j e_{j+1})^3 = \pm 1.$$

By substituting $e_j \to \pm e_j$, we may assume $(e_j e_{j+1})^3 = 1$. As in the proof of Lemma 1.1, note that if ψu, ψv $(u, v \in E)$ are conjugate elements of order 2 in S_n, then $u^2 = v^2$. It follows that e_j^2 and $(e_j e_k)^2$ $(|j-k| \geqslant 2)$ are constants independent of j and k. Hence, there are at most four double covers of S_n, corresponding to the choices of sign for e_j^2 and $(e_j e_k)^2$.

In case $e_j^2 = (e_j e_k)^2 = 1$, we obtain the group $E = \{\pm 1\} \times S_n$, which is not a representation group. In case $e_j^2 = \pm 1$, $(e_j e_k)^2 = -1$, we obtain the groups $\tilde{S}_n$ (take $\sigma_j = (-1)^j e_j$) and $\tilde{S}'_n$ (take $\sigma'_j = e_j$). Finally, if $e_j^2 = -1$, $(e_j e_k)^2 = 1$, we obtain a double cover that is not a representation group. From the substitution $e_j \to (-1)^j i e_j$, one finds that every spin representation of E is in this case projectively equivalent to a $\mathbf{C}S_n$-module. ∎

If $\sigma_j \mapsto A_j \in GL(V)$ defines a spin representation of $\tilde{S}_n$, then $\sigma'_j \mapsto iA_j$ defines a spin representation of $\tilde{S}'_n$, and conversely. Therefore, the representations of $\tilde{S}_n$ and $\tilde{S}'_n$ are essentially the same, even though the groups are isomorphic only when $n = 6$ [S, p. 166]. Following Schur, we will restrict our attention to the spin representations of $\tilde{S}_n$, rather than $\tilde{S}'_n$, for the remainder of this paper.

2. CONJUGACY CLASSES

The conjugacy classes of S_n are indexed by partitions $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_l)$ of n; $w \in S_n$ belongs to the λth conjugacy class if the cycle lengths of w are $\lambda_1, \lambda_2, \ldots, \lambda_l$. In such a case, we will write type$(w) = \lambda$, $\ell(\lambda) = l$, and $|\lambda| = n$. Designate λ to be *even* or *odd* according to whether $n - \ell(\lambda)$ is even or odd. The parity of λ is therefore the same as the parity of any permutation of type λ.

It is sometimes convenient to write $\lambda = (1^{m_1} 2^{m_2} \cdots)$ to indicate that m_j of the parts of λ are size j. In these terms, observe that

$$z_\lambda = (m_1! \, 1^{m_1})(m_2! \, 2^{m_2}) \cdots$$

is the size of the centralizer of any permutation of type λ.

For any $\sigma \in \tilde{S}_n$, let $|\sigma|$ denote the S_n-image of σ under the canonical homomorphism $\sigma_j \mapsto s_j$ and define

$$C_\lambda = \{\sigma \in \tilde{S}_n : \text{type } |\sigma| = \lambda\}.$$

Since the S_n-images of any $\sigma, \tau \in C_\lambda$ are S_n-conjugates, it follows that τ is conjugate to σ or $-\sigma$. Thus, if any $\sigma \in C_\lambda$ is conjugate to $-\sigma$, then C_λ is an $\tilde{S}_n$ conjugacy class. Otherwise, if there is no $\sigma \in C_\lambda$ conjugate to $-\sigma$, it follows that C_λ splits into two conjugacy classes namely,

$$C_\lambda^+ = \{\tau\sigma\tau^{-1} : \tau \in \tilde{S}_n\} \qquad \text{and} \qquad C_\lambda^- = \{-\tau\sigma\tau^{-1} : \tau \in \tilde{S}_n\}.$$

This labeling is not well defined, since it depends on the choice of σ. To resolve this ambiguity, we will choose a canonical representative σ^λ from each C_λ and designate C_λ^+ (for those C_λ's that split) to be the conjugacy class of σ^λ.

Specifically, we define

$$\sigma^\lambda = \pi_1 \pi_2 \cdots \pi_l \qquad (l = \ell(\lambda)), \tag{2.1}$$

where

$$\pi_j = \sigma_{a+1} \sigma_{a+2} \cdots \sigma_{a+\lambda_j - 1} \qquad (a = \lambda_1 + \cdots + \lambda_{j-1});$$

for example, when $\lambda = (3, 3, 1)$, we have $\sigma^\lambda = \sigma_1 \sigma_2 \sigma_4 \sigma_5$ and $|\sigma^\lambda| = (123)(456)(7)$. We remark that since distinct π_j's are products of anti-commuting σ_k's, it follows that

$$\pi_r \pi_s = (-1)^{(\lambda_r - 1)(\lambda_s - 1)} \pi_s \pi_r. \tag{2.2}$$

We will find that designating C_λ^+ to be the class of σ^λ is the "correct" choice, since the basic spin characters are positive on the even classes of the form C_λ^+ (Theorem 3.3).

The following result characterizes the split classes.

THEOREM 2.1 (Schur [S, p. 172]). *Let λ be a partition of n. Then C_λ splits into two $\tilde{S}_n$ conjugacy classes iff either* (1) *λ has only odd parts, or* (2) *λ has distinct parts and $n - \ell(\lambda)$ is odd.*

Since partitions of the types appearing in (1) and (2) play special roles in the study of spin characters of $\tilde{S}_n$, it is convenient to use the notation OP_n

to denote the partitions of n with only odd parts, and the notation DP_n to denote the partitions of n with distinct parts. Of course, every $\lambda \in OP_n$ is even, but the notation for DP_n needs to be further refined—let DP_n^+ and DP_n^- denote the even and odd partitions of DP_n, respectively.

Before proving Theorem 2.1, we note that as a corollary, the vector space of class functions spanned by spin characters (or equivalently, the center of $\mathbf{C}S'_n$) may be easily described. The space of all $\tilde{S}_n$ class functions is of the form $Z_n \oplus Z'_n$, where Z_n and Z'_n denote the spaces spanned by ordinary and spin characters, respectively. The indicator functions 1_λ defined by

$$1_\lambda(\sigma) = \begin{cases} 1 & \text{if type } |\sigma| = \lambda \\ 0 & \text{if type } |\sigma| \neq \lambda \end{cases}$$

are clearly a basis of Z_n. For the spin characters, define spin-indicator functions $1'_\lambda$ for each split class $C_\lambda^\pm$ via

$$1'_\lambda(\sigma) = \begin{cases} 1 & \text{if } \sigma \in C_\lambda^+ \\ -1 & \text{if } \sigma \in C_\lambda^- \\ 0 & \text{if type } |\sigma| \neq \lambda. \end{cases}$$

Note that if φ is the character of a spin representation $\tilde{S}_n \to GL(V)$, then $\varphi(\sigma) = -\varphi(-\sigma)$ since $-1 \in \tilde{S}_n$ is represented by $-1 \in GL(V)$. Therefore, φ vanishes on the conjugacy classes of $\tilde{S}_n$ that do not split, and we may conclude

COROLLARY 2.2. *$\{1'_\lambda : \lambda \in OP_n \cup DP_n^-\}$ is a basis of Z'_n.*

By our previous remarks, C_λ will split if and only if σ^λ and $-\sigma^\lambda$ are not conjugate. To prove Theorem 2.1, it therefore suffices to characterize the partitions λ for which the normalizer N_λ of $\pm\sigma^\lambda$ actually centralizes σ^λ.

As a first step, note that each π_j in (2.1) normalizes $\pm\sigma^\lambda$. In fact, a direct application of (2.2) proves the following:

LEMMA 2.3. *If $\pi = \pi_j$, then $\pi\sigma^\lambda\pi^{-1} = (-1)^{(\lambda_j - 1)(n - \ell(\lambda) - \lambda_j + 1)}\sigma^\lambda$.*

To investigate the remainder of N_λ, we need to choose a canonical $\tau \in \tilde{S}_n$ covering each transposition $t \in S_n$. Specifically, for the transposition $t = (pq)$ with $1 \leqslant p < q \leqslant n$, we choose

$$\tau_{pq} = \sigma_p \sigma_{p+1} \cdots \sigma_{q-2} \sigma_{q-1} \sigma_{q-2} \cdots \sigma_{p+1} \sigma_p \tag{2.3}$$

and use the abbreviation $\tilde{T} = \{\tau_{pq} : p < q\}$.

LEMMA 2.4. *Let $\tau \in \tilde{T}$ and $\sigma = \sigma_{j_1} \cdots \sigma_{j_l} \in \tilde{S}_n$. We have $\tau\sigma = (-1)^{l+s}\sigma\tau'$ for some $\tau' \in \tilde{T}$, where s is the number of solutions r $(0 \leqslant r < l)$ of*

$$\sigma_{j_1} \cdots \sigma_{j_r} \sigma_{j_{r+1}} \sigma_{j_r} \cdots \sigma_{j_1} = \pm\tau. \tag{2.4}$$

Proof. Proceed by induction on l. For $l=0$, the result is immediate. For $l>0$, we may write $\sigma=\sigma'\sigma_{j_l}$, where $\sigma'=\sigma_{j_1}\cdots\sigma_{j_{l-1}}$. By the induction hypothesis, we have $\tau\sigma=(\tau\sigma')\,\sigma_{j_l}=(-1)^{l-1+s'}\,\sigma'\tau'\sigma_{j_l}$ for some $\tau'\in\tilde{T}$, where s' is the number of solutions of (2.4) with $r<l-1$. Note that $\tau'=\pm(\sigma')^{-1}\,\tau\sigma'=\pm\sigma_{j_{l-1}}\cdots\sigma_{j_1}\tau\sigma_{j_1}\cdots\sigma_{j_{l-1}}$, and so we consider two cases: $\tau'=\sigma_{j_l}$ and $\tau'\neq\sigma_{j_l}$.

If $\tau'=\sigma_{j_l}$, then $s=s'+1$ and $\tau'\sigma_{j_l}=\sigma_{j_l}\tau'$. Therefore, $\tau\sigma=(-1)^{l+s}\,\sigma\tau'$, as desired.

If $\tau'\neq\sigma_{j_l}$, then we have $s=s'$ and we must prove $\tau'\sigma_{j_l}=-\sigma_{j_l}\tau''$ for some $\tau''\in\tilde{T}$ to produce the desired conclusion. That is, we must prove $\sigma_j\tau'\sigma_j\in\tilde{T}$ whenever $\tau'\neq\sigma_j$. To see this, suppose $\tau'=\tau_{pq}$. If $j=p$ or $p-1$, then $\sigma_j\tau_{pq}\sigma_j=\tau_{p+1,q}$ or $\tau_{p-1,q}$, respectively. If $j<p-1$ or $j>q$, then σ_j anticommutes with every term in (2.3), so $\sigma_j\tau_{pq}\sigma_j=\tau_{pq}$. If $p<j<q-1$, then we have

$$\begin{aligned}\sigma_j\tau_{pq}\sigma_j&=\sigma_p\cdots(\sigma_j\sigma_{j-1}\sigma_j\sigma_{j+1}\cdots\sigma_{q-1}\cdots\sigma_{j+1}\sigma_j\sigma_{j-1}\sigma_j)\cdots\sigma_p\\&=\sigma_p\cdots(\sigma_{j-1}\sigma_j\sigma_{j-1}\sigma_{j-1}\cdots\sigma_{q-1}\cdots\sigma_{j+1}\sigma_{j-1}\sigma_j\sigma_{j-1})\cdots\sigma_p=\tau_{pq}.\end{aligned}$$

Finally, if $j=q$ or $q-1$, we obtain $\sigma_j\tau_{pq}\sigma_j=\tau_{p,q+1}$ or $\tau_{p,q-1}$, by a similar argument. The cases $j=p$ and $j=q-1$ rely on the assumption $\tau'\neq\sigma_j$. ∎

LEMMA 2.5. *Suppose* $\lambda_j=\lambda_{j+1}=k$, *so that* $|\sigma^\lambda|$ *has two adjacent* k*-cycles; say* $(a+1,\ldots,a+k)(a+k+1,\ldots,a+2k)$. *If* $\pi=\pm\tau_1\cdots\tau_k$, *where* $\tau_j=\tau_{a+j,a+k+j}$, *then*

$$\pi\sigma^\lambda\pi^{-1}=(-1)^{k(n-\ell(\lambda))+k-1}\,\sigma^\lambda.$$

Proof. By inspection of S_n-images, it is easy to check that $\tau_j\sigma^\lambda=\pm\sigma^\lambda\tau_{j-1}$ (subscripts mod k). Apply Lemma 2.4 to τ_j and the factorization of σ^λ in (2.1). Since there are no solutions of (2.4) (again by inspection of S_n-images), we may conclude that $\tau_j\sigma^\lambda=(-1)^{n-\ell(\lambda)}\,\sigma^\lambda\tau_{j-1}$. By a further application of Lemma 2.4, one may verify that the τ_j's anticommute. Therefore,

$$\begin{aligned}\pi\sigma^\lambda\pi^{-1}&=(-1)^{k(n-\ell(\lambda))}\,\sigma^\lambda(\tau_k\tau_1\cdots\tau_{k-1})(\tau_k^{-1}\cdots\tau_2^{-1}\tau_1^{-1})\\&=(-1)^{k(n-\ell(\lambda))+k-1}\,\sigma^\lambda,\end{aligned}$$

as desired. ∎

Proof of Theorem 2.1. If $n-\ell(\lambda)$ is even and some λ_j is even, Lemma 2.3 shows that there exists $\pi\in N_\lambda$ such that $\pi\sigma^\lambda\pi^{-1}=-\sigma^\lambda$. If $n-\ell(\lambda)$ is odd, but λ has two repeated parts, Lemma 2.5 shows that there exists $\pi\in N_\lambda$ such that $\pi\sigma^\lambda\pi^{-1}=-\sigma^\lambda$. Hence, C_λ does not split in either of these cases.

Conversely, observe that N_λ is a double cover of the S_n-centralizer of $|\sigma^\lambda|$. Furthermore, the centralizer of any $w \in S_n$ is generated by the permutations w' that cyclically permute elements within cycles of w, or permute cycles of w of the same length. Consequently, the normalizer N_λ is generated by the elements π that appear in Lemmas 2.3 and 2.5. In case $\lambda \in OP_n$ or $\lambda \in DP_n^-$, the Lemmas show that these generators commute with σ^λ, and so C_λ splits. ∎

Next, we consider the conjugacy class of the parabolic subgroups of $\tilde{S}_n$. For any $J \subseteq \{1, 2, ..., n-1\}$, let $\tilde{S}_n^J$ denote the subgroup of $\tilde{S}_n$ generated by -1 and $\{\sigma_j : j \in J\}$. We call these parabolic subgroups since they are double covers of the more familiar parabolic S_n-subgroups $S_n^J = \langle s_j : j \in J \rangle$. Note that

$$S_n^J \cong S_{\beta_1} \times S_{\beta_2} \times \cdots \times S_{\beta_l}, \tag{2.5}$$

where $\beta = (\beta_1, ..., \beta_l)$ is the sequence of nonnegative integers of sum n defined indirectly by the condition

$$J = \{\beta_1 + \cdots + \beta_j : 1 \leqslant j < l\}.$$

In such a case, it will be convenient to write $S_n^J = S_\beta$, $\tilde{S}_n^J = \tilde{S}_\beta$, and let $\mathbf{C}S'_\beta$ denote the corresponding subalgebra of $\mathbf{C}S'_n$. By abuse of notation, we will regard S_{β_j} and $\tilde{S}_{\beta_j}$ as subgroups of S_β and $\tilde{S}_\beta$, respectively. For example, if $n = 9$ and $J = \{4, 6\}$, then $\beta = (4, 2, 3)$ and the subgroups $\tilde{S}_{\beta_1}$, $\tilde{S}_{\beta_2}$, $\tilde{S}_{\beta_3}$ are generated by $\{-1, \sigma_1, \sigma_2, \sigma_3\}$, $\{-1, \sigma_5\}$, and $\{-1, \sigma_7, \sigma_8\}$, respectively.

In view of (2.5), it is clear that the conjugacy classes of S_β are indexed by l-tuples $\boldsymbol{\lambda} = (\lambda^1, \lambda^2, ..., \lambda^l)$ in which λ^j is a partition of β_j. (The notation $\lambda^1, \lambda^2, ...$ refers to a sequence of partitions, whereas $\lambda_1, \lambda_2, ...$ refers to the parts of a single partition λ.)

Note that every $\sigma \in \tilde{S}_\beta$ may be factored nonuniquely in the form $\sigma = \pi_1 \cdots \pi_l$ with $\pi_j \in \tilde{S}_{\beta_j}$. By the same reasoning used for the conjugacy classes of $\tilde{S}_n$, it follows that

$$C_{\boldsymbol{\lambda}} = \{\pi_1 \cdots \pi_l : \pi_j \in \tilde{S}_{\beta_j}, \text{ type } |\pi_j| = \lambda^j\}$$

is either an $\tilde{S}_\beta$ conjugacy class, or it splits into two classes, according to whether any $\sigma \in C_{\boldsymbol{\lambda}}$ is $\tilde{S}_\beta$-conjugate to $-\sigma$. If $\sigma^{\lambda^j} \in \tilde{S}_{\beta_j}$ denotes the canonical representative (2.1) of C_{λ^j}, we designate

$$\sigma^{\boldsymbol{\lambda}} = \sigma^{\lambda^1} \cdots \sigma^{\lambda^l}$$

to be the canonical representative of $C_{\boldsymbol{\lambda}}$. That is, if $C_{\boldsymbol{\lambda}}$ splits into two conjugacy classes, let $C_{\boldsymbol{\lambda}}^+$ denote the conjugacy class of $\sigma^{\boldsymbol{\lambda}}$.

To characterize the split classes, fix a parabolic subgroup $\tilde{S}_\beta$ of $\tilde{S}_n$ and choose an l-tuple $\boldsymbol{\lambda} = (\lambda^1, ..., \lambda^l)$ with $|\lambda^j| = \beta_j$.

THEOREM 2.6. *Let $\lambda^* = \lambda^1 \cup \cdots \cup \lambda^l$ be the partition whose parts are the* *(multiset) union of those of* $\lambda^1, \ldots, \lambda^l$. *Then* C_λ *splits into two* $\tilde{S}_\beta$ *conjugacy classes iff either* (1) $\lambda^* \in OP_n$, *or* (2) λ^* *is odd and* $\lambda^j \in DP_{\beta_j}$ $(1 \leqslant j \leqslant l)$.

Proof. The centralizer of any $w = w_1 \cdots w_l \in S_\beta$ $(w_j \in S_{\beta_j})$ is the direct product of the S_{β_j}-centralizers of each w_j. Therefore, the $\tilde{S}_\beta$-normalizer N_λ of $\pm\sigma^\lambda$ is generated by those elements π that appear in Lemma 2.3, and those π in Lemma 2.5 that correspond to pairs of cycles in the same group $\tilde{S}_{\beta_j}$. Hence, C_λ splits if and only if $\pi\sigma^\lambda\pi^{-1} = \sigma^\lambda$ for all such π. Lemmas 2.3 and 2.5 easily imply that these conditions produce the desired conclusion. ∎

An element $\sigma \in \tilde{S}_n$ is said to be odd or even according to the parity of $|\sigma|$, and we let $\tilde{A}_n$ denote the subgroup of even elements. Note that $\tilde{A}_n$ forms a double cover of the alternating group; it is a representation group in all nontrivial cases except $n = 6$, 7 [S, p. 170].

An $\tilde{S}_n$ conjugacy class C of even elements is either an $\tilde{A}_n$ class, or will split into two such classes, according to whether the centralizer $C_{\tilde{S}_n}(\sigma)$ of any $\sigma \in C$ includes odd elements or not. In case C does split into two $\tilde{A}_n$ classes, say $C = C_+ \cup C_-$, then we have $C_+ = \sigma_1 C_- \sigma_1^{-1}$.

THEOREM 2.7 (Schur?). *Let* λ *be an even partition of* n. *The* $\tilde{S}_n$ *conjugacy class* C_λ *splits* (*if* $\lambda \in OP_n$ *then each of* $C_\lambda^\pm$ *split*) *into two* $\tilde{A}_n$ *classes iff* $\lambda \in DP_n^+$.

Proof. The centralizer $C_{\tilde{S}_n}(\sigma^\lambda)$ is a subgroup of the normalizer N_λ of $\pm\sigma^\lambda$. Recall from the proof of Theorem 2.1 that N_λ is generated by the elements $\pi \in \tilde{S}_n$ that appear in Lemmas 2.3 and 2.5.

First suppose that $\lambda \in DP_n^+$. Then there are no generators of the type in Lemma 2.5, and for those in Lemma 2.3, we have $\pi\sigma^\lambda\pi^{-1} = (\operatorname{sgn}|\pi|)\,\sigma^\lambda$. Hence, $C_{\tilde{S}_n}(\sigma^\lambda)$ contains only even elements, so C_λ (or $C_\lambda^\pm$) will not split into smaller $\tilde{A}_n$ classes.

Otherwise, we may suppose λ has repeated parts; say $\lambda_j = \lambda_{j+1} = k$. If k is odd, the corresponding element $\pi \in N_\lambda$ in Lemma 2.5 is odd, we have $\pi\sigma^\lambda\pi^{-1} = \sigma^\lambda$, and therefore, $C_{\tilde{S}_n}(\sigma^\lambda)$ contains odd elements. If k is even, then there is an even π such that $\pi\sigma^\lambda\pi^{-1} = -\sigma^\lambda$ in Lemma 2.5, but there is an odd π' in Lemma 2.3 such that $\pi'\sigma^\lambda(\pi')^{-1} = -\sigma^\lambda$. Hence, $\pi'\pi \in C_{\tilde{S}_n}(\sigma^\lambda)$, so in either case, $C_{\tilde{S}_n}(\sigma^\lambda)$ contains odd elements. ∎

3. Clifford Algebras and the Basic Spin Representation

The Clifford algebra $\mathscr{C}_n$ is a 2^n-dimensional algebra generated by n anticommuting involutions; i.e., $\mathscr{C}_n$ is the algebra generated by elements $\xi_1, \ldots, \xi_n$ with the presentation

$$\xi_j^2 = 1; \qquad \xi_j\xi_k = -\xi_k\xi_j \qquad (k \neq j). \tag{3.1}$$

As a basis of $\mathscr{C}_n$, we may take $\xi_A = \xi_{a_1} \cdots \xi_{a_r}$ where $A = \{a_1 < \cdots < a_r\}$ ranges over all subsets of $\{1, 2, ..., n\}$.

The spin representations of $\tilde{S}_n$ are deeply connected to the representation of Clifford algebras. In fact, we will see that $\tilde{S}_n$ may be realized as a subgroup of (the multiplicative group of) a Clifford algebra, so that the restriction of any Clifford module to $\tilde{S}_n$ yields a spin representation. In particular, the smallest faithful representation of $\tilde{S}_n$, known as the basic spin representation, can be constructed this way.

We note that $\mathscr{C}_n$ is a twisted group algebra for $\mathbf{Z}_2^n$, and therefore semisimple. When n is even, $\mathscr{C}_n$ is actually simple. Perhaps the most concrete way to prove this is to explicitly exhibit an isomorphism $\rho\colon \mathscr{C}_n \to M_{2^k}$ between $\mathscr{C}_n$ and the full matrix algebra $M_{2^k} \cong M_2^{\otimes k}$ of order $2^k (n = 2k)$. One such example, taken from [W, VIII.13], is given by

$$\begin{aligned} \xi_{2j-1} &\overset{\rho}{\longmapsto} \varepsilon \otimes \cdots \otimes \varepsilon \otimes x \otimes 1 \otimes \cdots \otimes 1 \\ \xi_{2j} &\overset{\rho}{\longmapsto} \varepsilon \otimes \cdots \otimes \varepsilon \otimes y \otimes 1 \otimes \cdots \otimes 1, \end{aligned} \tag{3.2}$$

where x and y occur in the jth position $(1 \leqslant j \leqslant k)$, and

$$\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

One needs to check that the defining relations (3.1) are satisfied, and that the resulting algebra morphism is surjective. The reader may consult [W] for details.

The character of ρ is easily calculated. The M_{2^k}-image of ξ_A is a tensor product of k elements taken from $\{1, \varepsilon, x, y\}$. Of these elements, only 1 has nonzero trace. Hence, for any coefficients $c_A \in \mathbf{C}$, we have

$$\operatorname{tr} \rho \left(\sum_A c_A \xi_A \right) = 2^k c_\phi.$$

Since $\mathscr{C}_{2k} \cong M_{2^k}$, this is the unique irreducible character of $\mathscr{C}_{2k}$.

Next consider the algebra $\mathscr{C}_{2k+1}$. Observe that $\zeta = \xi_1 \cdots \xi_{2k+1}$ belongs to the center $Z\mathscr{C}_{2k+1}$, and $\zeta^2 = (-1)^k$. By Schur's lemma, it follows that in any irreducible representation of $\mathscr{C}_{2k+1}$, ζ must be represented by the scalar $\pm i^k$. Given the representation $\rho\colon \mathscr{C}_{2k} \to M_{2^k}$ defined by (3.2), extend ρ to two representations $\rho_\pm\colon \mathscr{C}_{2k+1} \to M_{2^k}$ by defining $\rho_\pm(\zeta) = \pm i^k$. The maps $\rho_\pm$ are well defined since $\mathscr{C}_{2k+1} = \mathscr{C}_{2k} \oplus \zeta \mathscr{C}_{2k}$ as vector spaces, and they are algebra morphisms since ζ is central. Hence, $\mathscr{C}_{2k+1} \cong M_{2^k} \oplus M_{2^k}$ as algebras. Furthermore, note that $\operatorname{tr} \rho_\pm(\zeta \xi_A) = 0$ for $A \subset \{1, 2, ..., 2k\}$, unless $A = \varnothing$. In that case, we have $\operatorname{tr} \rho_\pm(\zeta \xi_\phi) = \operatorname{tr} \rho_\pm(\zeta) = \pm(2i)^k$.

In summary, we have the following well-known result:

PROPOSITION 3.1. (a) *The unique irreducible character of* $\mathscr{C}_{2k} \cong M_{2^k}$ *is given by*

$$\operatorname{tr} \rho \left(\sum_A c_A \xi_A \right) = 2^k c_\phi.$$

(b) *The two irreducible characters of* $\mathscr{C}_{2k+1} \cong M_{2^k} \oplus M_{2^k}$ *are given by*

$$\operatorname{tr} \rho_\pm \left(\sum_A c_A \xi_A \right) = 2^k c_\phi \pm (2i)^k c_\zeta,$$

where c_ζ *denotes the coefficient of* $\zeta = \xi_1 \cdots \xi_{2k+1}$.

Clifford algebras can be used to resolve the problem we postponed in Section 1, i.e., establishing the existence of a factor set corresponding to the twisted algebra $\mathbf{C}S_n'$, or equivalently, proving that the relations (1.4) define a group of order $2 \cdot n!$.

In view of the Coxeter presentation of S_n in (1.2), it is clear that S_n is a homomorphic image of $\tilde{S}_n$, and therefore we must have either $|\tilde{S}_n| = n!$ or $2 \cdot n!$. Thus, it suffices to show that $1 = -1$ is not a consequence of (1.4). This can be proved by exhibiting a homomorphic image of $\tilde{S}_n$ in which $1 \neq -1$. In fact, we claim that the homomorphism $\tilde{S}_n \to \mathscr{C}_n^*$ defined by

$$\sigma_j \mapsto \frac{i}{\sqrt{2}} (\xi_j - \xi_{j+1}) \tag{3.3}$$

satisfies our requirements. The proof is complete as soon as one checks that the defining relations (1.4) are satisfied; however, the patient reader may prefer to apply the more general Lemma 3.2 below. From the point of view of twisted algebras, the map $\alpha_j \mapsto (\xi_j - \xi_{j+1})/\sqrt{2}$ defines an injection $\mathbf{C}S_n' \to \mathscr{C}_n$.

Now that $\tilde{S}_n$ has been realized as a subgroup of $\mathscr{C}_n^*$, we are free to apply the representations of $\mathscr{C}_n$ described above to obtain spin representations of $\tilde{S}_n$. Unfortunately, these fail to be irreducible when n is even. One way to circumvent this difficulty, aside from explicitly decomposing the resulting $\mathbf{C}S_n'$-modules, is to realize $\tilde{S}_n$ as a subgroup of $\mathscr{C}_{n-1}^*$. The following lemma characterizes the possible embeddings $\tilde{S}_n \to \mathscr{C}_m^*$ in which the σ_j's are represented linearly.

LEMMA 3.2. *Let* $A = [a_{jk}]$ *be an* $(n-1) \times m$ *matrix. The map* $\sigma_j \mapsto \sum_k a_{jk} \xi_k$ *defines a homomorphism* $\tilde{S}_n \to \mathscr{C}_m^*$ *if*

$$AA' = \begin{bmatrix} -1 & 1/2 & & 0 \\ 1/2 & -1 & \ddots & \\ & \ddots & \ddots & 1/2 \\ 0 & & 1/2 & -1 \end{bmatrix}. \tag{3.4}$$

The converse holds only if $n \neq 3$.

Proof. For an arbitrary A, define $\bar{\sigma}_j = \sum_k a_{jk} \xi_k$ $(1 \leqslant j < n)$. We have

$$\bar{\sigma}_j \bar{\sigma}_k + \bar{\sigma}_k \bar{\sigma}_j = 2 \sum_r a_{jr} a_{kr} = 2(AA')_{jk} \qquad (1 \leqslant j, k < n). \tag{3.5}$$

Therefore, $\bar{\sigma}_j^2 = -1$ iff $(AA')_{jj} = -1$, and $\bar{\sigma}_j \bar{\sigma}_k = -\bar{\sigma}_k \bar{\sigma}_j$ iff $(AA')_{jk} = 0$. Furthermore, if we assume $\bar{\sigma}_j^2 = \bar{\sigma}_{j+1}^2 = -1$ and let $\bar{\sigma}_j \bar{\sigma}_{j+1} + \bar{\sigma}_{j+1} \bar{\sigma}_j = a \in \mathbf{C}$, then we have

$$\begin{aligned} \bar{\sigma}_j \bar{\sigma}_{j+1} \bar{\sigma}_j - \bar{\sigma}_{j+1} \bar{\sigma}_j \bar{\sigma}_{j+1} &= \bar{\sigma}_j (a - \bar{\sigma}_j \bar{\sigma}_{j+1}) - \bar{\sigma}_{j+1} (a - \bar{\sigma}_{j+1} \bar{\sigma}_j) \\ &= (a-1)(\bar{\sigma}_j - \bar{\sigma}_{j+1}). \end{aligned} \tag{3.6}$$

If A satisfies (3.4), then we have $a = 1$, and so $(\bar{\sigma}_j \bar{\sigma}_{j+1})^3 = -1$. Thus $\sigma_j \mapsto \bar{\sigma}_j$ does define an $\tilde{S}_n$-homomorphism.

To prove the converse, assume $n \geqslant 4$ and suppose that $\sigma_j \mapsto \bar{\sigma}_j$ defines an $\tilde{S}_n$-homomorphism. From the above calculations, it is sufficient to verify that $(AA')_{j,j+1} = \frac{1}{2}$. Note that the relation $(\bar{\sigma}_j \bar{\sigma}_{j+1})^3 = -1$ shows that (3.6) can be satisfied only if $(AA')_{j,j+1} = \frac{1}{2}$ or $\bar{\sigma}_j = \bar{\sigma}_{j+1}$. However, if $\bar{\sigma}_1 = \bar{\sigma}_2$, for example, then we find $(\bar{\sigma}_1 \bar{\sigma}_3)^3 = (\bar{\sigma}_1 \bar{\sigma}_3)^2 = -1$, a contradiction. Hence, A must satisfy (3.4).

If $n = 3$, there are additional solutions satisfying $AA' = \left[\begin{smallmatrix} -1 & -1 \\ -1 & -1 \end{smallmatrix}\right]$. ∎

For example, Lemma 3.2 shows that

$$\sigma_j \mapsto a_j \xi_j + b_j \xi_{j+1} \tag{3.7}$$

defines a homomorphism $\tilde{S}_n \to \mathscr{C}_n^*$ for any choice of $a_j, b_j \in \mathbf{C}$ $(1 \leqslant j < n)$ satisfying $a_j^2 + b_j^2 = -1$ and $a_{j+1} b_j = \frac{1}{2}$. One such example appears in (3.3). If we choose $b_{n-1} = 0$, we obtain an embedding $\tilde{S}_n \to \mathscr{C}_{n-1}^*$.

For any homomorphism $\psi: \tilde{S}_n \to \mathscr{C}_{n-1}^*$ of the type in Lemma 3.2, define the *basic spin representation(s)* with respect to ψ to be the representation(s) of $\tilde{S}_n$ obtained from either the composition

$$\tilde{S}_n \xrightarrow{\psi} \mathscr{C}_{n-1}^* \xrightarrow{\rho} GL_{2^k} \qquad (n = 2k+1)$$

or both of the compositions

$$\tilde{S}_n \xrightarrow{\psi} \mathscr{C}_{n-1}^* \xrightarrow{\rho\pm} GL_{2^{k-1}} \qquad (n = 2k).$$

These representations will be shown to be independent of ψ, modulo equivalence. In particular, the basic spin characters, denoted by φ^n (n odd) and $\varphi^n_\pm$ (n even), are well defined. Under any circumstances in which n is even and the choice of φ^n_+ or φ^n_- is irrelevant, we will use the notation φ^n.

THEOREM 3.3 (Schur [S, p. 203]). *The basic spin representations of $\tilde{S}_n$ are irreducible. Furthermore, we have*

$$\varphi^{2k+1}(\sigma^\lambda) = 2^{(l(\lambda)-1)/2} \qquad \text{if } \lambda \in OP_{2k+1}$$

$$\varphi^{2k}_\pm(\sigma^\lambda) = \begin{cases} 2^{(l(\lambda)-2)/2} & \text{if } \lambda \in OP_{2k} \\ \pm i^k\sqrt{k} & \text{if } \lambda = (2k), \end{cases}$$

and in all other cases, $\varphi^n(\sigma^\lambda) = 0$.

Proof. First, we verify that the basic spin representations are independent of the choice of ψ, up to equivalence. Given two choices A, $A' \in GL_{n-1}$ satisfying (3.4), we have $AA^t = A'A'^t$, so $A^{-1}A'$ is orthogonal; i.e., there is a (complex) orthogonal transformation O such that $A' = AO$. Since the orthogonal group acts on $\mathscr{C}_{n-1}$ via $\xi_j \mapsto \xi'_j = \sum_k o_{jk}\xi_k$ (cf. (3.5) when A is orthogonal), it follows that the automorphism $\xi_j \mapsto \xi'_j$ induces an automorphism $\rho(\xi_j) \mapsto \rho(\xi'_j)$ of the simple algebra M_{2^k}, where $n = 2k+1$ or $n = 2k+2$. Since $\mathrm{Aut}(M_{2^k}) = GL_{2^k}$, acting by conjugation, we may conclude that the basic spin representations produced by A and A' are equivalent.

To determine the characters, we are now free to choose the embedding $\tilde{S}_n \to \mathscr{C}^*_{n-1}$ in (3.7) with $b_{n-1} = 0$; it is uniquely determined by (3.4) aside from choices of square roots.

Let $\bar{\sigma}^\lambda$ denote the image of σ^λ in $\mathscr{C}^*_{n-1}$. Proposition 3.1 implies that $\varphi^n(\sigma^\lambda)$ depends only on the coefficients c_ϕ and c_ζ of 1 and ζ in $\bar{\sigma}^\lambda$, respectively. By definition (2.1), we have $\bar{\sigma}^\lambda = \bar{\pi}_1 \cdots \bar{\pi}_l$, where $\bar{\pi}_j$ is of the form

$$(a_r\xi_r + b_r\xi_{r+1})(a_{r+1}\xi_{r+1} + b_{r+1}\xi_{r+2}) \cdots (a_s\xi_s + b_s\xi_{s+1}). \tag{3.8}$$

Since each of $\xi_1, \ldots, \xi_{n-1}$ appear in at most one of the $\bar{\pi}_j$'s, the coefficient of 1 is the product of the coefficients of 1 in each $\bar{\pi}_j$. But there is no constant term in (3.8) unless there is an even number of factors. Therefore, $c_\phi = 0$ unless $\lambda \in OP_n$. In that case, the constant term of $\bar{\pi}_j$ in (3.8) is

$$b_r a_{r+1} b_{r+1} a_{r+2} \cdots b_{s-1} a_s = (1/2)^{(\lambda_j - 1)/2},$$

and therefore $c_\phi = 2^{(\ell(\lambda)-n)/2}$.

In case $n = 2k$ and the coefficient c_ζ is nonzero, there must be at least $2k-1$ factors in $\bar{\sigma}^\lambda$. This can only happen if $\lambda = (2k)$, and in that case,

$$\bar{\sigma}^\lambda = (a_1\xi_1 + b_1\xi_2) \cdots (a_{2k-1}\xi_{2k-1} + 0\xi_{2k}).$$

Therefore, $c_\zeta = a_1 a_2 \cdots a_{2k-1} = \det A$ when $\lambda = (2k)$. A routine calculation (cf. (3.4)) shows that $\det A^2 = -2^{-(2k-2)}k$, so $c_\zeta = \pm i 2^{-(k-1)}\sqrt{k}$. The claimed formulas for $\varphi^n(\sigma^\lambda)$ are now an immediate consequence of Proposition 3.1 and our formulas for c_ϕ and c_ζ.

Finally, we must prove that the basic spin representations are irreducible. This can be verified by showing that φ^n is of unit length in the character metric of $\tilde{S}_n$.

Recall from Section 2 that z_λ denotes the size of the S_n-centralizer of any $w \in S_n$ of type λ. Therefore, if $C_\lambda^\pm$ is any pair of split $\tilde{S}_n$-classes, we have $|C_\lambda^+| = |C_\lambda^-| = n!/z_\lambda$. It follows that

$$\|\varphi^n\|^2 = \frac{1}{|\tilde{S}_n|} \sum_{\sigma \in \tilde{S}_n} |\varphi^n(\sigma)|^2 = \sum_{\lambda \in OP_n} \frac{1}{z_\lambda} |\varphi^n(\sigma^\lambda)|^2 + (0 \text{ or } 1/2),$$

where the alternative 0 or $\frac{1}{2}$ occurs in the cases $n = 2k+1$ and $n = 2k$, respectively. From the formulas for $\varphi^n(\sigma^\lambda)(\lambda \in OP_n)$, we see that to prove $\|\varphi^n\|^2 = 1$ it will suffice to show

$$\sum_{\lambda \in OP_n} \frac{1}{z_\lambda} 2^{\ell(\lambda)} = 2.$$

To prove this identity, let $u_1, u_2, \ldots$ be indeterminates, and use the abbreviation $u_\lambda = u_{\lambda_1} u_{\lambda_2} \cdots$ for any partition λ. The cycle indicator of S_n is the coefficient of $t^n/n!$ in the generating function [Ma, I(2.14)]

$$\sum_\lambda \frac{t^{|\lambda|}}{z_\lambda} u_\lambda = \exp\left(\sum_{r \geq 1} \frac{t^r}{r} u_r \right). \tag{3.9}$$

Under the substitution $u_{2r} \to 0$, $u_{2r+1} \to 2$, we obtain the desired expansion:

$$\sum_{\lambda \in OP} \frac{t^{|\lambda|}}{z_\lambda} 2^{\ell(\lambda)} = \frac{1+t}{1-t} = 1 + 2t + 2t^2 + \cdots. \quad \blacksquare$$

We remark that Morris [Mo1] constructs the basic spin characters by a different method. Rather than realizing $\tilde{S}_n$ as a subgroup of a Clifford algebra, Morris regards S_n as a subgroup of the orthogonal group O_n (via permutation matrices), and restricts the (projective) basic spin representation of O_n to S_n. This fails to be irreducible for even n, and so does not, strictly speaking, give an explicit construction of the basic spin representations of $\tilde{S}_{2k}$. One way to circumvent this annoyance is to restrict the basic spin representation of O_{n-1} to the reflection representation of S_n.

4. Induced Products for Spin Representations

Fix a parabolic subgroup $\tilde{S}_n^J = \tilde{S}_\beta$, and write $J = \bigcup_{k=1}^l J_k$, where

$$J_k = \{ j \colon \beta_1 + \cdots + \beta_{k-1} < j < \beta_1 + \cdots + \beta_k \}.$$

Continuing the notational abuse of Section 2, regard $\tilde{S}_{\beta_k}$ as the subgroup of $\tilde{S}_\beta$ generated by -1 and $\{\sigma_j : j \in J_k\}$.

In order to induce spin representations from $\tilde{S}_\beta$ to $\tilde{S}_n$, one first needs techniques for constructing spin representations of $\tilde{S}_\beta$, or equivalently, $\mathbf{C}S'_\beta$-modules. One solution to this problem may be described as follows. Let $V_1, ..., V_l$ be modules for $\mathbf{C}S'_{\beta_1}, ..., \mathbf{C}S'_{\beta_l}$, and let V be a module for the Clifford algebra $\mathscr{C}_l$. The tensor product $V \otimes V_1 \otimes \cdots \otimes V_l$ may be given a $\mathbf{C}S'_\beta$-module structure by defining

$$\sigma_j(v \otimes v_1 \otimes \cdots \otimes v_l) = \xi_k v \otimes v_1 \otimes \cdots \otimes \sigma_j v_k \otimes \cdots \otimes v_l \qquad (j \in J_k), \tag{4.1}$$

and we call this the *Clifford product* of $V_1, ..., V_l$ with respect to V. To prove that this does define a $\mathbf{C}S'_\beta$-module, one needs to check that the defining relations are satisfied; i.e., $\sigma_r^2 = -1$ for $r \in J$, $(\sigma_r \sigma_s)^2 = -1$ for $r, s \in J(|r-s| \geqslant 2)$, and $(\sigma_r \sigma_{r+1})^3 = -1$ when r, $r+1 \in J_k$. We leave this easy exercise to the reader.

By inducing Clifford products from $\tilde{S}_\beta$ to $\tilde{S}_n$, we obtain a simple procedure for creating spin representations of $\tilde{S}_n$ from spin representations of the $\tilde{S}_{\beta_k}$'s. The analogous procedure for S_n (the outer tensor product) is much simpler, but is still a powerful tool in the construction and analysis of irreducible $\mathbf{C}S_n$-modules. However, unlike the outer tensor product, the Clifford product has the unfortunate property that even if the modules V_j (and V) are irreducible, their Clifford product need not be irreducible. Typically, a Clifford product is a (large) multiple of one or two irreducible $\mathbf{C}S'_\beta$-modules. In such a case, none of the constituents of the induced $\mathbf{C}S'_n$-module would be multiplicity-free, and so it provides little useful information about irreducible $\mathbf{C}S'_n$-moules.

To avoid this problem, we will construct a more elaborate operation we call the *reduced Clifford product* that yields irreducible $\mathbf{C}S'_\beta$-modules when the modules V_j are irreducible. To describe this operation, we will first require a digression into the internal structure of $\mathbf{C}S'_n$-modules.

Two $\mathbf{C}S'_n$-modules V, V' are said to be *associates* if their characters φ, φ' satisfy $\varphi(\sigma) = \operatorname{sgn} |\sigma| \cdot \varphi'(\sigma)$; i.e., $V' \cong \operatorname{sgn} \otimes V$. For example, the basic spin representation of $\tilde{S}_{2k+1}$ is self-associate, whereas the two basic spin representations of $\tilde{S}_{2k}$ are associates.

Let V be an irreducible self-associate $\mathbf{C}S'_n$-module. Since $V \cong \operatorname{sgn} \otimes V$, there must exist $S \in GL(V)$ so that

$$S\sigma_j|_V = -\sigma_j S|_V \qquad (1 \leqslant j < n). \tag{4.2}$$

We call S an *associator* for V. Since S^2 commutes with the action of $\tilde{S}_n$, Schur's lemma implies that S^2 is a scalar. We will always assume that this scalar is chosen so that $S^2 = 1$. Schur's lemma also shows that S is unique up to scalar multiples, so this convention uniquely determines S up to a factor of ± 1. Define the *difference character* $\Delta\colon \tilde{S}_n \to \mathbf{C}$ of V via

$$\Delta(\sigma) = \operatorname{tr}(S\sigma \mid_V).$$

Observe that Δ is uniquely determined, aside from the fact that $-S$ produces the difference character $-\Delta$.

The relations between the irreducible representations of a group G and those of any subgroup of index 2 are well known. The following result summarizes the relations for the pair $(\tilde{S}_n, \tilde{A}_n)$. We have included a proof for the sake of completeness.

LEMMA 4.1. *Let V be an irreducible $\mathbf{C}S'_n$-module with character φ.*

(a) *If V is self-associate, then the restriction $V \downarrow \tilde{A}_n$ $(n \geqslant 2)$ is of the form $V^+ \oplus V^-$, where $V^\pm$ are inequivalent, irreducible $\tilde{A}_n$-modules. Moreover,*

$$\Delta(\sigma) = \operatorname{tr}(\sigma \mid_{V^+}) - \operatorname{tr}(\sigma \mid_{V^-}) \qquad (\sigma \in \tilde{A}_n).$$

(b) *If V is not self-associate, then $V \downarrow \tilde{A}_n$ is irreducible.*

Proof. For (a), let S be an associator for V. Since $S^2 = 1$, the eigenvalues of S are ± 1, and $V = V^+ \oplus V^-$, where $V^\pm = \{v \in V\colon Sv = \pm v\}$. Since S commutes with the action of $\tilde{A}_n$ but is not a scalar (cf. (4.2)), we conclude that V^+ and V^- are nontrivial $\tilde{A}_n$-modules. The claimed trace formula is now immediate. To see that $V^\pm$ is irreducible, note that $\varphi(\sigma) = 0$ unless $\sigma \in \tilde{A}_n$, by (4.2). Applying the character metric, we see that $\|\varphi\|^2_{\tilde{A}_n} = 2\, \|\varphi\|^2_{\tilde{S}_n} = 2$. This can happen only if V^+ and V^- are irreducible and inequivalent.

For (b), let $\varphi' = \operatorname{sgn} \cdot \varphi$ be the associate of φ. Observe that $\varphi \downarrow \tilde{A}_n = \varphi' \downarrow \tilde{A}_n$ and $\varphi(\sigma) + \varphi'(\sigma) = 0$ unless $\sigma \in \tilde{A}_n$. Therefore, we have

$$4\, \|\varphi\|^2_{\tilde{A}_n} = \|\varphi + \varphi'\|^2_{\tilde{A}_n} = 2\, \|\varphi + \varphi'\|^2_{\tilde{S}_n} = 4,$$

since φ and φ' are $\tilde{S}_n$-orthogonal. This can only happen if $\|\varphi\|^2_{\tilde{A}_n} = 1$; i.e., $V \downarrow \tilde{A}_n$ is irreducible. ∎

We are now ready to define the reduced Clifford product. Let $V_1, \ldots, V_l$ be modules for $\mathbf{C}S'_{\beta_1}, \ldots, \mathbf{C}S'_{\beta_l}$, of which exactly r are self-associate and s have inequivalent associates $(l = r + s)$. For simplicity, we assume that $V_1, \ldots, V_r$ are the self-associate modules. Choose any (irreducible) module V for the Clifford algebra $\mathscr{C}_s$ generated by $\{\xi_{r+1}, \ldots, \xi_l\}$, and choose associators S_j for each V_j $(1 \leqslant j \leqslant r)$. (We can always find involutions $S_j \in GL(V_j)$ satisfying (4.2), but we cannot assert that $\pm S_j$ is unique unless V_j is irreducible). The *reduced Clifford product of* $V_1, \ldots, V_l$ with respect to

V and $S_1, ..., S_r$ is the tensor product $V \otimes V_1 \otimes \cdots \otimes V_l$, with $\mathbf{C}S'_\beta$-module structure defined by

$$\sigma_j(v \otimes v_1 \otimes \cdots \otimes v_l) = \begin{cases} v \otimes A_1 v_1 \otimes \cdots \otimes A_l v_l & \text{if } j \in J_k,\ 1 \leqslant k \leqslant r \\ \xi_k v \otimes B_1 v_1 \otimes \cdots \otimes B_l v_l & \text{if } j \in J_k,\ r < k \leqslant l, \end{cases} \tag{4.3}$$

where the operators A and B are chosen as follows:

$$(A_1, ..., A_l) = (S_1, ..., S_{k-1}, \sigma_j, 1, ..., 1)$$
$$(B_1, ..., B_l) = (S_1, ..., S_r, 1, ..., \sigma_j, ..., 1).$$

In either case, note that σ_j $(j \in J_k)$ occurs in the kth position. The fact that this does define a $\mathbf{C}S'_\beta$-module is an easy consequence of (1.4) and (4.2).

Proposition 4.2. *Let $\varphi_j = \operatorname{char} V_j$, and let $\pi_j \in \tilde{S}_{\beta_j}$. The character φ of the reduced Clifford product* (4.3) *may be described as follows.*

(a) *If every π_j is even, then*

$$\varphi(\pi_1 \cdots \pi_l) = 2^{\lfloor s/2 \rfloor} \varphi_1(\pi_1) \cdots \varphi_l(\pi_l).$$

(b) *If s is odd and Δ_j is the difference character of V_j, then*

$$\varphi(\pi_1 \cdots \pi_l) = \pm (2i)^{\lfloor s/2 \rfloor} \Delta_1(\pi_1) \cdots \Delta_r(\pi_r)\, \varphi_{r+1}(\pi_{r+1}) \cdots \varphi_l(\pi_l),$$

provided that π_j is even for $1 \leqslant j \leqslant r$ and odd for $r < j \leqslant l$.

(c) *In all other cases, $\varphi(\pi_1 \cdots \pi_l) = 0$.*

Proof. Observe that the action of $\pi' = \pi_{r+1} \cdots \pi_l$ on $W := V \otimes V_{r+1} \otimes \cdots \otimes V_l$ is identical to the action of π' on the unreduced product as in (4.1). In particular, we have

$$\pi'(v \otimes v_{r+1} \otimes \cdots \otimes v_l) = \xi_A v \otimes \pi_{r+1} v_{r+1} \otimes \cdots \otimes \pi_l v_l,$$

where $A = \{j : \pi_j \text{ is odd}\}$. By Proposition 3.1, there are only two possible cases for which $v \mapsto \xi_A v$ will have nonzero trace—either $A = \varnothing$, or s is odd and $A = \{r+1, ..., l\}$.

In the case $A = \varnothing$, we have π_j even for $r < j \leqslant l$, and

$$\operatorname{tr}(\pi' \mid_W) = 2^{\lfloor s/2 \rfloor} \varphi_{r+1}(\pi_{r+1}) \cdots \varphi_l(\pi_l).$$

Furthermore, if we identify $W \otimes V_1 \otimes \cdots \otimes V_r$ with the reduced Clifford product, (4.3) implies that the action of $\pi = \pi_1 \cdots \pi_l$ may be written in the equivalent form

$$\pi(w \otimes v_1 \otimes \cdots \otimes v_r) = \pi' w \otimes S_1^{m_1} \pi_1 v_1 \otimes \cdots \otimes S_r^{m_r} \pi_r v_r,$$

where $(-1)^{m_j} = \operatorname{sgn} |\pi_{j+1} \cdots \pi_r|$. In particular, $m_r = 0 \pmod 2$. Since V_r is self-associate, we have $\varphi_r(\pi_r) = 0$ unless π_r is even. Therefore, $\varphi(\pi) = 0$ unless $(-1)^{m_{r-1}} = \operatorname{sgn} |\pi_r| = 1$; i.e., $m_{r-1} = 0 \pmod 2$. By induction on j, it follows that $\varphi(\pi) = 0$ unless $m_j = 0 \pmod 2$ and every π_j is even $(1 \leqslant j \leqslant r)$. The trace formula claimed in (a) now follows.

In the case for which s is odd and $A = \{r+1, ..., l\}$, we have π_j odd for $r < j \leqslant l$, and Proposition 3.1 implies

$$\operatorname{tr}(\pi' \mid_W) = \pm (2i)^{\lfloor s/2 \rfloor} \varphi_{r+1}(\pi_{r+1}) \cdots \varphi_l(\pi_l).$$

Furthermore, (4.3) implies

$$\pi(w \otimes v_1 \otimes \cdots \otimes v_r) = \pi' w \otimes S_1^{m_1} \pi_1 S_1 v_1 \otimes \cdots \otimes S_r^{m_r} \pi_r S_r v_r.$$

Since $m_r = 0 \pmod 2$, we have $\operatorname{tr}(S_r^{m_r} \pi_r S_r \mid_{V_r}) = 0$ unless π_r is even (cf. (4.2)). Again by induction on j, it is easy to see that $\varphi(\pi) = 0$ unless $m_j = 0 \pmod 2$ and every π_j is even $(1 \leqslant j \leqslant r)$. In that case,

$$\pi(w \otimes v_1 \otimes \cdots \otimes \otimes v_r) = (-1)^r \pi' w \otimes S_1 \pi_1 v_1 \otimes \cdots \otimes S_r \pi_r v_r,$$

from which (b) follows. ∎

We remark that the $\pm$ sign that appears in (b) depends only on the choice of V and $S_1, ..., S_r$. It does not depend on the choice of $\pi = \pi_1 \cdots \pi_l$.

The following result provides evidence that the reduced Clifford product deserves to be considered the spin analogue of the outer tensor product.

THEOREM 4.3. *A reduced Clifford product of irreducible $\mathbf{C}S'_{\beta_j}$-modules is irreducible. Conversely, every irreducible $\mathbf{C}S'_{\beta}$-module is of this form.*

Proof. Let $V_1, ..., V_l$ be irreducible modules for $\mathbf{C}S'_{\beta_1}, ..., \mathbf{C}S'_{\beta_l}$, and assume for simplicity that $V_1, ..., V_r$ are the self-associate modules, so that we may use the notation and character formulas of Proposition 4.2. To prove that the reduced Clifford product is irreducible, we will verify that its character φ is of unit length in the character metric of $\tilde{S}_\beta$.

If s is even, Proposition 4.2 implies that $\|\varphi\|^2 = 2^{s-l} a_1 \cdots a_l$, where $a_j = \|\varphi_j \downarrow \tilde{A}_{\beta_j}\|^2$. By Lemma 4.1, we have $a_j = 2$ for $1 \leqslant j \leqslant r$ and $a_j = 1$ for $r < j \leqslant l$, so $\|\varphi\| = 1$, as desired.

If s is odd, Proposition 4.2 implies $\|\varphi\|^2 = \|\varphi\|^2 = 2^{s-1-l} a_1 \cdots a_l + 2^{s-1-l} b_1 \cdots b_r c_{r+1} \cdots c_l$, where $b_j = \|\Delta_j \downarrow \tilde{A}_{\beta_j}\|^2$ and

$$c_j = \frac{1}{\beta_j!} \sum_{\sigma \notin \tilde{A}_{\beta_j}} |\varphi_j(\sigma)|^2 = 2 \|\varphi_j\|^2 - \|\varphi_j \downarrow \tilde{A}_{\beta_j}\|^2$$

By Lemma 4.1, we have $b_j = 2$ $(1 \leqslant j \leqslant r)$ and $c_j = 1$ $(r < j \leqslant l)$. Hence, we have $\|\varphi\| = 1$ in either case, so the reduced Clifford product is irreducible.

To prove that every irreducible $\mathbf{C}S'_\beta$-module is a reduced Clifford product, we will show that the number of such products we have constructed is correct; i.e., we claim that the number of these products is the same as the number of classes C_λ of $\tilde{S}_\beta$ that split.

LEMMA 4.4. *The number of self-associate (resp., pairs of inequivalent associate) irreducible spin characters of* $\tilde{S}_n$ *is* $|DP_n^+|$ *(resp.,* $|DP_n^-|$*).*

Proof. We have $Z'_n = Z_n^+ \oplus Z_n^-$, where Z_n^+ (resp., Z_n^-) denotes the subspace of Z'_n spanned by class functions that vanish outside the even (resp., odd) classes. By Corollary 2.2, Z_n^+ (resp., Z_n^-) has a basis indexed by OP_n (resp., DP_n^-).

If φ is any self-associate spin character, then $\varphi \in Z_n^+$, whereas if φ and φ' are a pair of inequivalent associate spin characters, then $\varphi + \varphi' \in Z_n^+$ and $\varphi - \varphi' \in Z_n^-$. Therefore

$$|OP_n| = \dim Z_n^+ = s_n + a_n \qquad \text{and} \qquad |DP_n^-| = \dim Z_n^- = a_n,$$

where s_n and a_n denote the number of irreducible self-associate and associate pairs of spin characters, respectively. To complete the lemma, recall that $|DP_n| = |OP_n|$ (e.g., see [A, Chap. 1]), so we have $s_n = |OP_n| - |DP_n^-| = |DP_n^+|$. ∎

This lemma shows that the reduced Clifford products of irreducible $\mathbf{C}S'_{\beta_j}$-modules may be labeled by l-tuples $\boldsymbol{\lambda} = (\lambda^1, ..., \lambda^l)$ of partitions with $\lambda^j \in DP_{\beta_j}$. In case the number of λ^j of odd sign (i.e., the number of λ^j that label non-self-associate modules) is odd, we have constructed two products corresponding to $\boldsymbol{\lambda}$. Since there is always at least one product corresponding to each choice of $\boldsymbol{\lambda}$, we may apply the fact that $|DP_{\beta_j}| = |OP_{\beta_j}|$ to account for each split $\tilde{S}_\beta$-class that appears in case (1) of Theorem 2.6. The remaining products, one each corresponding to those $\boldsymbol{\lambda}$ with an odd number of odd λ^j's, are accounted for precisely by case (2) of Theorem 2.6. This completes the proof of Theorem 4.3. ∎

At this point, we have the basic spin representations available for contructing Clifford products. However, we cannot explicitly construct, nor can we compute the character of, a reduced Clifford product of these representations unless we can find the associator of the basic spin representation of $\tilde{S}_{2k+1}$.

In Section 3, we realized $\tilde{S}_{2k+1}$ as a subgroup of $\mathscr{C}^*_{2k}$ and obtained the basic spin representation via the composition $\tilde{S}_{2k+1} \to \mathscr{C}^*_{2k} \to^{\rho} GL_{2^k}$. For the purposes of computation, it is easier to realize $\tilde{S}_{2k+1}$ as a subgroup of $\mathscr{C}^*_{2k+1}$ via (3.3) and use the composition $\tilde{S}_{2k+1} \to \mathscr{C}^*_{2k+1} \to^{\rho_\pm} GL_{2^k}$. We leave to the reader the easy task of verifying that the character of this representation is indeed φ^{2k+1}, regardless of the choice of sign in $\rho_\pm$.

To construct the associator for this representation, consider

$$\xi = \frac{1}{\sqrt{2k+1}} (\xi_1 + \cdots + \xi_{2k+1}) \in \mathscr{C}^*_{2k+1}.$$

Observe that $\xi^2 = 1$ and $\xi(\xi_j - \xi_{j+1}) = -(\xi_j - \xi_{j+1})\xi$, so ξ anticommutes with the $\mathscr{C}^*_{2k+1}$-image of σ_j in (3.3). It follows that $\rho_\pm(\xi)$ is the associator.

LEMMA 4.5. *Let Δ^{2k+1} be the difference character of φ^{2k+1}. We have*

$$\Delta^{2k+1}(\sigma^\lambda) = \pm i^k \sqrt{2k+1} \qquad \text{if } \lambda = (2k+1),$$

and in all other cases, $\Delta^{2k+1}(\sigma^\lambda) = 0$.

Proof. Let $\bar{\sigma}^\lambda$ denote the $\mathscr{C}^*_{2k+1}$-image of σ^λ. Since $\Delta^{2k+1}(\sigma) = 0$ unless σ is even, we may assume λ is even. In that case, (2.1) and (3.3) imply that $\xi\bar{\sigma}^\lambda$ is a product of an odd number of linear factors, and therefore has no constant term. By Proposition 3.1, it follows that $\Delta^{2k+1}(\sigma^\lambda) = \pm(2i)^k c_\zeta$, where c_ζ is the coefficient of ζ in $\xi\bar{\sigma}^\lambda$. This coefficient will vanish unless there are at least $2k+1$ linear factors in $\xi\bar{\sigma}^\lambda$. This happens only when $\lambda = (2k+1)$, and in that case,

$$\xi\bar{\sigma}^\lambda = \frac{1}{\sqrt{2k+1}} \left(\frac{i}{\sqrt{2}}\right)^{2k} (\xi_1 + \cdots + \xi_{2k+1})(\xi_1 - \xi_2) \cdots (\xi_{2k} - \xi_{2k+1}).$$

Extracting the coefficient of ζ yields

$$\Delta^{2k+1}(\sigma^\lambda) = \pm(2i)^k \cdot \frac{1}{\sqrt{2k+1}} \left(\frac{i}{\sqrt{2}}\right)^{2k} (2k+1) = \pm i^k \sqrt{2k+1},$$

as claimed. ∎

For any partition λ of n with $\ell(\lambda) = l$, let R^λ denote a reduced Clifford product of the basic spin representations of $\tilde{S}_{\lambda_1}, ..., \tilde{S}_{\lambda_l}$, and let θ^λ denote the $\tilde{S}_\lambda$-character of R^λ. If λ is odd, then the number of factors not self-associate is odd, so there are two possible products indexed by λ. In circumstances where the choice between these two products is significant, we will use the notations $R^\lambda_\pm$ and $\theta^\lambda_\pm$.

Using Theorems 3.3 and Proposition 4.2, it is easy to give character formulas for θ^λ on even classes. For the odd classes, Proposition 4.2 and Lemma 4.5 show that $\theta^\lambda(\sigma) = 0$ unless λ is odd and σ is an $\tilde{S}_\lambda$-conjugate of $\pm\sigma^\lambda$. In that case, we find

$$\theta^\lambda_\pm(\sigma^\lambda) = \pm i^{(n-l+1)/2} \sqrt{\tfrac{1}{2}\lambda_1 \cdots \lambda_l}. \tag{4.4}$$

5. A Characteristic Map for Spin Characters

Let $\Lambda = \bigoplus_{n \geqslant 0} \Lambda^n$ denote the (graded) ring of symmetric functions in the variables $x_1, x_2, \ldots$ with coefficients in $\mathbf{Z}$. It will sometimes be convenient to enlarge the coefficient ring to a field F of characteristic 0. In such a case, we will use the notation Λ_F and regard Λ_F as an F-algebra.

For a partition λ, the sum m_λ of all distinct permutations of the monomial $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots$ is called a *monomial symmetric function.* Clearly, $\{m_\lambda : |\lambda| = n\}$ forms a $\mathbf{Z}$-basis of Λ^n.

The *power-sum symmetric function* p_r is defined for $r > 0$ by

$$p_r = x_1^r + x_2^r + \cdots,$$

and we use the abbreviation $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$ for any partition λ. It is well known that the p_r's are algebraically independent generators of $\Lambda_{\mathbf{Q}}$, so that, in particular, $\{p_\lambda : |\lambda| = n\}$ forms a basis of $\Lambda_{\mathbf{Q}}^n$.

A third basis of the ring Λ that we will need to consider is formed by the *Schur functions* s_λ. The reader is referred to [Ma] for their definition and a proof of the fact that $\{s_\lambda : |\lambda| = n\}$ is a $\mathbf{Z}$-basis of Λ^n.

Let $\{u_\lambda : |\lambda| = n\}$ and $\{v_\lambda : |\lambda| = n\}$ be an arbitrary pair of bases of Λ_F^n (or Λ^n), and define a nondegenerate bilinear form B on Λ_F^n by setting $B(u_\lambda, v_\mu) = \delta_{\lambda\mu}$. Over the complex field, use forms that are conjugate-linear in the second variable (i.e., $B(f, cg) = \bar{c}B(f, g)$) so that B is an inner product when $u_\lambda = v_\lambda$. Introduce a new set of indeterminates $y_1, y_2, \ldots$, and use $u_\lambda(x)$ and $v_\lambda(y)$ as abbreviations for $u_\lambda(x_1, x_2, \ldots)$ and $v_\lambda(y_1, y_2, \ldots)$. It is easy to check that the form B depends on u_λ and v_λ only via the generating function

$$f_B(x, y) = \sum_{|\lambda| = n} u_\lambda(x)\, \bar{v}_\lambda(y),$$

where $\bar{v}$ indicates coefficient conjugation. That is, any other pair of bases will yield the same generating function $f_B(x, y)$ if and only if they define the same form. (This observation is implicit in [Ma, I.4] and Lemma 2.1 of [St]). We may thus regard f_B as a *definition* of the form B.

To recover the form defined by a generating function $f(x, y)$, choose any basis $\{u_\lambda : |\lambda| = n\}$ of Λ_F^n, and let $v_\lambda(y)$ be the coefficient of $u_\lambda(x)$ in $f(x, y)$. Assuming that f did arise from a nondegenerate form, then $\{v_\lambda : |\lambda| = n\}$ must also be a basis of Λ^n, and u_λ is dual to $\bar{v}_\lambda$.

Similar considerations apply to bilinear forms defined on any finite dimensional subspace of Λ_F. They also apply to infinite dimensional graded subspaces of Λ_F, provided that the bases considered are homogeneous.

The prototypical illustration of this point of view is provided by the bilinear form $\langle\ ,\ \rangle$ defined on Λ by the generating function [Ma, I.4]

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda \frac{1}{z_\lambda} p_\lambda(x)\, p_\lambda(y) = \sum_\lambda s_\lambda(x)\, s_\lambda(y).$$

It follows that

$$\frac{1}{z_\lambda}\langle p_\lambda, p_\mu\rangle = \langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu},$$

and $\langle\ ,\ \rangle$ is an inner product on Λ_F.

The *characteristic map* ch: $Z_n \to \Lambda^n_{\mathbf{C}}$ is the linear isomorphism defined by $\mathrm{ch}(1_\lambda) = z_\lambda^{-1} p_\lambda$. Since $\langle 1_\lambda, 1_\mu\rangle_{S_n} = z_\lambda^{-1}\delta_{\lambda\mu}$ in the character metric of S_n, it follows that ch is an isometry; i.e.,

$$\langle \chi, \theta\rangle_{S_n} = \langle \mathrm{ch}\,\chi, \mathrm{ch}\,\theta\rangle$$

for any χ, $\theta \in Z_n$. We remark that it can be shown (see [Ma, I.7]) that multiplication in Λ corresponds to the induction of outer tensor products; i.e.,

$$\mathrm{ch}(\chi)\,\mathrm{ch}(\theta) = \mathrm{ch}((\chi \times \theta)\uparrow S_n), \tag{5.1}$$

where $\chi \times \theta$ denotes the outer tensor product of an S_k-character χ and an S_{n-k}-character θ. Frobenius constructed the irreducible characters $\{\chi^\lambda : |\lambda| = n\}$ of S_n by proving that $\chi^\lambda = \mathrm{ch}^{-1}(s_\lambda)$, from which it follows that $\langle s_\lambda, p_\mu\rangle$ is the character table of S_n.

We now consider the problem of developing an analogue of ch for spin characters that duplicates, as well as possible, all of the above properties of the characteristic map.

Let $\Omega_F = \bigoplus_{n\geqslant 0}\Omega^n_F$ denote the graded subalgebra of Λ_F generated by 1, $p_1, p_3, p_5, \ldots,$ and let $\Omega = \Omega_{\mathbf{Q}} \cap \Lambda$ denote the **Z**-coefficient (graded) subring of $\Omega_{\mathbf{Q}}$. Clearly, $\{p_\lambda : \lambda \in OP_n\}$ forms a basis of $\Omega^n_{\mathbf{Q}}$. The *spin characteristic* $\mathrm{ch}' : Z'_n \to \Omega^n_{\mathbf{C}}$, is defined to be the linear map given by

$$\mathrm{ch}'(1'_\lambda) = \begin{cases} z_\lambda^{-1} 2^{\ell(\lambda)/2} p_\lambda & \text{if } \lambda \in OP_n \\ 0 & \text{if } \lambda \in DP_n^-. \end{cases}$$

Note that $\mathrm{ch}'(\varphi) = 0$ iff $\varphi \downarrow \tilde{A}_n = 0$.

Define an inner product $[\ ,\ ]$ on $\Omega^n_{\mathbf{C}}$ by setting

$$[p_\lambda, p_\mu] = z_\lambda 2^{-\ell(\lambda)}\delta_{\lambda\mu} \qquad (\lambda, \mu \in OP_n). \tag{5.2}$$

Since $\langle 1'_\lambda, 1'_\mu\rangle_{\tilde{S}_n} = \frac{1}{2}\langle 1'_\lambda, 1'_\mu\rangle_{\tilde{A}_n} = z_\lambda^{-1}\delta_{\lambda\mu}$ for λ, $\mu \in OP_n$, we see that ch′ essentially preserves the $\tilde{A}_n$ character metric; i.e.,

PROPOSITION 5.1. *$[\mathrm{ch}'\varphi, \mathrm{ch}'\theta] = \frac{1}{2}\langle\varphi, \theta\rangle_{\tilde{A}_n}$ for any φ, $\theta \in Z'_n$.*

From the identity (3.9) and the specialization

$$u_r \to \begin{cases} 2p_r(x)\,p_r(y) & \text{if } r \text{ odd} \\ 0 & \text{if } r \text{ even,} \end{cases}$$

one may deduce

PROPOSITION 5.2. *$[\ ,\ ]$ is the bilinear form defined by the generating function*

$$\prod_{i,j} \frac{1+x_i y_j}{1-x_i y_j} = \sum_{\lambda \in OP} \frac{1}{z_\lambda} 2^{\ell(\lambda)} p_\lambda(x)\, p_\lambda(y).$$

THEOREM 5.3. *Let $\varphi_1, ..., \varphi_l$ be spin characters, of which s are not self-associate, and let $\varphi_1 \times_c \cdots \times_c \varphi_l$ denote their reduced Clifford product. We have*

$$\mathrm{ch}'((\varphi_1 \times_c \cdots \times_c \varphi_l) \uparrow \tilde{S}_n) = 2^{\lfloor s/2 \rfloor} \mathrm{ch}'(\varphi_1) \cdots \mathrm{ch}'(\varphi_l).$$

Proof. The following argument parallels Macdonald's proof of (5.1).

Assume that φ_j is a spin character of $\tilde{S}_{\beta_j}$, and by the usual abuse of notation, regard the $\tilde{S}_{\beta_j}$'s as subgroups of the parabolic subgroup $\tilde{S}_\beta$. Define

$$\psi_n = \sum_{\lambda \in OP_n} 2^{\ell(\lambda)/2} p_\lambda 1'_\lambda,$$

and regard ψ_n as an $\Omega^n_{\mathbb{C}}$-valued class function of $\tilde{S}_n$.

LEMMA 5.4. *$\psi_n(\pi_1 \cdots \pi_l) = \psi_{\beta_1}(\pi_1) \cdots \psi_{\beta_l}$ for any $\pi_j \in \tilde{S}_{\beta_j}$.*

Proof. Both $\psi_n(\pi_1 \cdots \pi_l)$ and $\psi_{\beta_1}(\pi_1) \cdots \psi_{\beta_l}(\pi_l)$ are $\tilde{S}_\beta$-class functions that vanish unless every π_j is even. By Theorem 2.6, it is therefore sufficient to choose an l-tuple $\boldsymbol{\lambda} = (\lambda^1, ..., \lambda^l)(\lambda^j \in OP_{\beta_j})$, and consider the case $\sigma^{\boldsymbol{\lambda}} = \pi_1 \cdots \pi_l$; i.e., $\pi_j = \sigma^{\lambda^j}$.

Since $\mathrm{type}|\sigma^{\boldsymbol{\lambda}}| = \lambda^1 \cup \cdots \cup \lambda^l = \lambda^*$, we have $\sigma^{\boldsymbol{\lambda}} \in C^{\pm}_{\lambda^*}$. Therefore,

$$\psi_{\beta_1}(\sigma^{\lambda^1}) \cdots \psi_{\beta_l}(\sigma^{\lambda^l}) = 2^{\ell(\lambda^*)/2} p_{\lambda^*} = \pm \psi_n(\sigma^{\boldsymbol{\lambda}}),$$

and to complete the lemma, we must prove $\sigma^{\boldsymbol{\lambda}} \in C^+_{\lambda^*}$. By Theorem 3.3, it is sufficient to show $\varphi^n(\sigma^{\boldsymbol{\lambda}}) > 0$. From the proof of this theorem (particularly (3.8)), it is clear that the constant term of the $\mathscr{C}^*_{n-1}$-image of $\sigma^{\boldsymbol{\lambda}}$ depends only on λ^*, and not on any particular ordering of the parts of $\boldsymbol{\lambda}$. Thus, $\varphi^n(\sigma^{\boldsymbol{\lambda}}) = \varphi^n(\sigma^{\lambda^*}) > 0$, as desired. ∎

To complete the theorem, observe that from the definitions of ψ_n and the spin characteristic, we have $\mathrm{ch}'\varphi = \langle \varphi, \psi_n \rangle_{\tilde{S}_n}$ for any $\varphi \in Z'_n$. Hence, by Frobenius reciprocity,

$$\mathrm{ch}'((\varphi_1 \times_c \cdots \times_c \varphi_l) \uparrow \tilde{S}_n) = \langle \varphi_1 \times_c \cdots \times_c \varphi_l, \psi_n \downarrow \tilde{S}_\beta \rangle$$

so by Lemma 5.4 and Proposition 4.2

$$\begin{aligned} &= 2^{-l} \langle 2^{\lfloor s/2 \rfloor} \varphi_1 \cdots \varphi_l, \psi_{\beta_1} \cdots \psi_{\beta_l} \rangle_{\tilde{A}_{\beta_1} \times \cdots \times \tilde{A}_{\beta_l}} \\ &= 2^{\lfloor s/2 \rfloor} (\tfrac{1}{2} \langle \varphi_1, \psi_{\beta_1} \rangle_{\tilde{A}_{\beta_1}}) \cdots (\tfrac{1}{2} \langle \varphi_l, \psi_{\beta_l} \rangle_{\tilde{A}_{\beta_l}}) \\ &= 2^{\lfloor s/2 \rfloor} \mathrm{ch}'(\varphi_1) \cdots \mathrm{ch}'(\varphi_l). \quad ∎ \end{aligned}$$

As an application, we calculate the spin characteristics of the basic spin characters φ^n and the induced characters $\theta^\lambda \uparrow \tilde{S}_n$ of the reduced Clifford products of Section 4.

Define symmetric functions $q_n \in \Lambda^n$ via the generating function

$$\sum_{n \geqslant 0} q_n t^n = \prod_j \frac{1 + x_j t}{1 - x_j t}, \tag{5.3}$$

and use the abbreviation $q_\lambda = q_{\lambda_1} q_{\lambda_2} \cdots$ for any partition λ. From Proposition 5.2 and the specialization $y_1 \to t$, $y_2, y_3, \ldots \to 0$, we obtain

$$q_n = \sum_{\lambda \in OP_n} \frac{1}{z_\lambda} 2^{\ell(\lambda)} p_\lambda. \tag{5.4}$$

Therefore, $q_n \in \Omega^n$ and $\{q_\lambda : \lambda \in OP_n\}$ is a basis of Ω^n_Q. By Theorem 3.3, it follows that

$$\mathrm{ch}'(\varphi^n) = \frac{1}{\varepsilon_n \sqrt{2}} q_n, \tag{5.5}$$

where $\varepsilon_{2k} = \sqrt{2}$ and $\varepsilon_{2k+1} = 1$.

For the reduced Clifford product $\theta^\lambda = \varphi^{\lambda_1} \times_c \cdots \times_c \varphi^{\lambda_l}$, assume that s of the parts of λ are even. By Theorem 5.3, we have

$$\mathrm{ch}'(\theta^\lambda \uparrow \tilde{S}_n) = 2^{\lfloor s/2 \rfloor - (l+s)/2} q_\lambda.$$

Using the notation

$$\varepsilon_\lambda = \begin{cases} \sqrt{2} & \text{if } \lambda \text{ odd} \\ 1 & \text{if } \lambda \text{ even,} \end{cases} \tag{5.6}$$

we may record this as follows:

Proposition 5.5. $\mathrm{ch}'(\theta^\lambda \uparrow \tilde{S}_n) = \varepsilon_\lambda^{-1} 2^{-\ell(\lambda)/2} q_\lambda$.

The symmetric functions q_λ are particularly useful with respect to [,], as the following result shows.

Proposition 5.6. (a) *We have* (cf. [Ma, III(4.2)])

$$\prod_{i,j} \frac{1 + x_i y_j}{1 - x_i y_j} = \sum_\lambda q_\lambda(x)\, m_\lambda(y).$$

(b) *If $f \in \Omega_{\mathbf{C}}$, then $[f, q_\lambda]$ is the coefficient of m_λ in f.*

$$\begin{array}{cccccc} 1 & 1 & 1 & 2 & 2 & 8 \\ 2 & 3 & 4 & 4 & & \\ 3 & 4 & 6 & 6 & & \\ 6 & & & & & \end{array} \qquad \begin{array}{cccccc} 1' & 1 & 1 & 2' & 2 & 5' \\ & 2 & 2 & 2 & 5' & 7 \\ & & 3' & 5' & 5 & \\ & & & 5' & & \end{array}$$

(a) Ordinary (b) Shifted

FIGURE 1

Proof. By (5.3), we have

$$\prod_{i,j} \frac{1+x_i y_j}{1-x_i y_j} = \left(\sum_{\beta_1 \geqslant 0} q_{\beta_1}(x)\, y_1^{\beta_1} \right) \left(\sum_{\beta_2 \geqslant 0} q_{\beta_2}(x)\, y_2^{\beta_2} \right) \cdots,$$

from which it is clear that the coefficient of $m_\lambda(y)$ is $q_\lambda(x)$.

For (b) it is sufficient to consider $f = p_\mu$ ($\mu \in OP_n$). Let $a_{\mu\lambda}$ denote the coefficient of m_λ in p_μ. If we extract the coefficient of $m_\lambda(y)$ from the identities in part (a) and Proposition 5.2, we obtain

$$q_\lambda = \sum_{\mu \in OP_n} \frac{1}{z_\mu} 2^{\ell(\mu)} a_{\mu\lambda} p_\mu.$$

Since the coefficients $a_{\mu\lambda}$ are real, (5.2) implies $[p_\mu, q_\lambda] = a_{\mu\lambda}$, as desired. ∎

6. SHIFTED TABLEAUX AND SCHUR'S Q-FUNCTIONS

Each partition λ has associated with it a *diagram*

$$D_\lambda = \{(i, j) \in \mathbf{Z}^2 \colon 1 \leqslant j \leqslant \lambda_i,\ 1 \leqslant i \leqslant \ell(\lambda)\}.$$

The elements of D_λ are viewed (by Anglophiles) as boxes in a plane with matrix-style coordinates.

An *ordinary tableau* T of *shape* λ is an assignment $T \colon D_\lambda \to \mathbf{P}$ of letters from the ordered alphabet $\mathbf{P} = \{1 < 2 < 3 < \cdots\}$ satisfying

(1) $T(i, j) < T(i+1, j)$ (increasing columns)

(2) $T(i, j) \leqslant T(i, j+1)$ (nondecreasing rows).

An example appears in Fig. 1a. Given such a tableau, let γ_k denote the number of boxes $(i, j) \in D_\lambda$ such that $T(i, j) = k$. The tableau T is said to have *content* $\gamma = (\gamma_1, \gamma_2, \ldots)$, and we will write $x^T = x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \cdots$.

The combinatorial theory of tableaux and the theory of symmetric functions overlap considerably. This is illustrated by the well-known fact that

$$s_\lambda(x) = \sum_{T:\, D_\lambda \to \mathbf{P}} x^T = \sum_\mu K_{\lambda\mu} m_\mu(x), \tag{6.1}$$

summed over tableaux T and partitions μ, where $K_{\lambda\mu}$ denotes the number of ordinary tableaux of shape λ and content μ. From this point of view, the existence of Knuth's correspondence [K] is a manifestation of the orthonormality of the irreducible S_n-characters. Conversely, it is not difficult to take (6.1) as the definition of s_λ, and use the combinatorial theory of tableaux to recover Frobenius' description of the irreducible characters.

In this and the following section, we will show that it is similarly possible to use the combinatorial theory of shifted tableaux to recover Schur's description of the irreducible spin characters.

Let $\mathbf{P}'$ denote the ordered alphabet $\{1' < 1 < 2' < 2 < \cdots\}$. The letters $1', 2', 3', \ldots$ are said to be *marked*, and we use the notation $|a|$ to refer to the unmarked version of any $a \in \mathbf{P}'$.

For each $\lambda \in DP$ there is an associated *shifted diagram* defined via

$$D'_\lambda = \{(i, j) \in \mathbf{Z}^2 \colon i \leqslant j \leqslant \lambda_i + i - 1,\ 1 \leqslant i \leqslant \ell(\lambda)\},$$

and a *shifted tableau T of shape λ* is an assignment $T\colon D'_\lambda \to \mathbf{P}'$ satisfying

$$\begin{aligned} &(1)\quad T(i,j) \leqslant T(i+1,j),\ T(i,j) \leqslant T(i,j+1);\\ &(2)\quad \text{Each column has at most one } k\ (k = 1, 2, \ldots);\\ &(3)\quad \text{Each row has at most one } k'\ (k' = 1', 2', \ldots). \end{aligned} \tag{6.2}$$

An example appears in Fig. 1b. Given such a tableau, let γ_k denote the number of boxes $(i, j) \in D'_\lambda$ such that $|T(i, j)| = k$. The tableau T is said to have *content* $\gamma = (\gamma_1, \gamma_2, \ldots)$, and we will write $x^T = x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \cdots$.

Define generating functions $Q_\lambda = Q_\lambda(x)$ in the variables $x_1, x_2, \ldots$ for each $\lambda \in DP$ via

$$Q_\lambda(x) = \sum_{T:\, D'_\lambda \to \mathbf{P}'} x^T, \tag{6.3}$$

summed over shifted tableaux T. As noted in the Introduction, it is far from obvious that these generating functions coincide with the symmetric polynomials defined by Schur [S, p. 225], but we will never need to make use of this equivalence in our derivation. In fact, it is not even obvious that (6.3) defines a symmetric function—this is proved in Corollary 6.2 below.

An alternative formulation of (6.3) that is sometimes convenient may be obtained by counting the number of ways to mark an assignment $T: D'_\lambda \to \mathbf{P}$ of unmarked letters so that the result is a shifted tableau. The only feasible unmarked assignments of this type satisfy property (1) of (6.2) and the property $T(i, j) < T(i+1, j+1)$ (i.e., increasing diagonals). The marking of a given entry $k = T(i, j)$ of a feasible T is indeterminate if and only if $T(i-1, j) \neq k$ and $T(i, j+1) \neq k$. In such a case, we will refer to the (i, j)-entry of T (or any associated shifted tableau obtained by marking T) as *free*, and let $\mathrm{fr}(T)$ denote the number of free entries of T. The example in Fig. 1b has $\mathrm{fr}(T) = 6$.

Since the markings of the free entries of T can be assigned independently and arbitrarily, it follows that

$$Q_\lambda(x) = \sum_{T: D'_\lambda \to \mathbf{P}} 2^{\mathrm{fr}(T)} x^T, \tag{6.4}$$

summed over feasible T. Since every entry on the main diagonal of a feasible T is free, every coefficient in Q_λ is divisible by $2^{\ell(\lambda)}$. Therefore,

$$P_\lambda(x) := 2^{-\ell(\lambda)} Q_\lambda(x) = \sum_{T: D'_\lambda \to \mathbf{P}'}{}' \; x^T$$

defines a formal power series with integer coefficients, where the notation $\sum'$ indicates that the sum is restricted to shifted tableaux with unmarked main diagonals.

The following result, due independently to Sagan and Worley, is a shifted analogue of Knuth's correspondence. The reader may consult [Sa] or [Wo] for a proof.

THEOREM 6.1. *Let A be a nonnegative integer matrix whose positive entries are marked arbitrarily, and let (S, T) be an ordered pair of shifted tableaux of the same shape in which the main diagonal of T is unmarked. There is a natural bijection between the matrices A and the tableaux pairs (S, T) such that:*

(1) *If S has constant β, then A has column sum vector β.*

(2) *If T has content γ, then A has row sum vector γ.*

The properties of this bijection are summarized in the generating function identity

$$\prod_{i,j} \frac{1 + x_i y_j}{1 - x_i y_j} = \sum_{\lambda \in DP} Q_\lambda(x) P_\lambda(y). \tag{6.5}$$

COROLLARY 6.2. (a) *The P_λ's and Q_λ's are symmetric functions.*

(b) $\{P_\lambda : \lambda \in DP_n\}$ *is a* **Z**-*basis of* Ω^n.

(c) $[P_\lambda, Q_\mu] = \delta_{\lambda\mu}$.

(d) *We have*

$$q_\mu = \sum_{\lambda \in DP_n} K'_{\lambda\mu} P_\lambda \qquad (|\mu| = n) \tag{6.6}$$

$$Q_\lambda = \sum_{|\mu| = n} K'_{\lambda\mu} m_\mu \qquad (\lambda \in DP_n), \tag{6.7}$$

where $K'_{\lambda\mu}$ is the number of shifted tableaux of shape λ and content μ.

Before proving this corollary, we need to establish that the (rectangular) matrix $[K'_{\lambda\mu}]$ is triangular with respect to the partial order on partitions of n defined by

$$\lambda \geqslant \mu \qquad \text{iff } \lambda_1 + \cdots + \lambda_j \geqslant \mu_1 + \cdots + \mu_j \text{ for all } j \geqslant 1.$$

LEMMA 6.3. *Let $\lambda \in DP_n$, $|\mu| = n$.*

(a) $K'_{\lambda\mu} = 0$ *unless* $\lambda \geqslant \mu$.

(b) $[2^{-\ell(\lambda)} K'_{\lambda\mu}]$ *is an integer matrix with a unit diagonal.*

Proof. Let T be a shifted tableau of shape λ and content μ. Recall that $|T|$ has increasing diagonals. Therefore, the $\mu_1 + \cdots + \mu_j$ entries of $|T|$ that are $\leqslant j$ must occur among the $\lambda_1 + \cdots + \lambda_j$ boxes in the first j rows; i.e., $\lambda \geqslant \mu$.

For (b), observe that in the case $\lambda = \mu$, the jth row of $|T|$ must consist entirely of j's. Therefore, the only free entries of T are on the main diagonal, and we have $K'_{\lambda\lambda} = 2^{\ell(\lambda)}$. We know that $2^{-\ell(\lambda)} K'_{\lambda\mu} \in \mathbf{Z}$ since it is the coefficient of x^μ in P_λ. ∎

Proof of Corollary 6.2. By Proposition 5.6 and (6.5), we have

$$\sum_{|\mu| = n} m_\mu(x)\, q_\mu(y) = \sum_{\lambda \in DP_n} Q_\lambda(x)\, P_\lambda(y).$$

The identity (6.6) is obtained by extracting the coefficient of the monomial x^μ. Since Lemma 6.3 shows that the matrix $[K'_{\lambda\mu}](\lambda, \mu \in DP_n)$ is invertible over **Q**, we may deduce (a) from (6.6). Now that Q_λ is known to be symmetric, (6.7) may be viewed as a restatement of the definition in (6.3).

The invertibility of (6.6) also proves $P_\lambda, Q_\lambda \in \Omega_{\mathbf{Q}}$, and Lemma 6.3 shows that the P_λ's and Q_λ's are linearly independent. Since $\dim \Omega^n_{\mathbf{Q}} = |OP_n| = |DP_n|$, it follows that $\{P_\lambda : \lambda \in DP_n\}$ and $\{Q_\lambda : \lambda \in DP_n\}$ are both bases of $\Omega^n_{\mathbf{Q}}$. Part (c) is now a consequence of (6.5) and Proposition 5.2.

To prove (b), let $f \in \Omega$. We know that there is an expansion of the form $f = \sum c_\lambda P_\lambda$ for some $c_\lambda \in \mathbf{Q}$. However, Lemma 6.3 and (6.7) imply that $P_\lambda = m_\lambda +$ lower terms (in the partial order $>$), so the coefficients c_λ are all $\mathbf{Z}$-linear combinations of the monomial coefficients in f; i.e., $c_\lambda \in \mathbf{Z}$. ∎

7. The Irreducible Spin Characters of $\tilde{S}_n$ and $\tilde{A}_n$

In describing the irreducible spin characters it will be convenient to define

$$Q_\lambda^* = 2^{-\ell(\lambda)/2} Q_\lambda = 2^{\ell(\lambda)/2} P_\lambda$$

for each $\lambda \in DP$. By Corollary 6.2(c), the Q_λ^*'s are an orthonormal basis of $\Omega_\mathbf{R}$.

Define a self-associate class function $\varphi^\lambda \in Z_n'$ for each $\lambda \in DP_n^+$ via

$$\varphi^\lambda(\sigma^\mu) = [Q_\lambda^*, 2^{\ell(\mu)/2} p_\mu] \qquad \text{if } \mu \in OP_n,$$

and define a pair of associate class functions $\varphi_\pm^\lambda \in Z_n'$ for each $\lambda \in DP_n^-$ via

$$\varphi_\pm^\lambda(\sigma^\mu) = \begin{cases} (1/\sqrt{2})[Q_\lambda^*, 2^{\ell(\mu)/2} p_\mu] & \text{if } \mu \in OP_n \\ \pm i^{(n-\ell(\lambda)+1)/2} \sqrt{\tfrac{1}{2} z_\lambda} & \text{if } \mu = \lambda. \end{cases}$$

For all other choices of μ, we define $\varphi^\lambda(\sigma^\mu) = 0$. When λ is odd, the notation φ^λ will be used to refer to either φ_+^λ or φ_-^λ.

Using the ε-notation defined in (5.6), the restriction of the φ^λ's to the even classes may be summarized in the relations

$$2^{\ell(\mu)/2} p_\mu = \sum_{\lambda \in DP_n} \varphi^\lambda(\sigma^\mu)\, \varepsilon_\lambda Q_\lambda^*.$$

This expansion may be inverted via (5.2), yielding

$$\frac{1}{\varepsilon_\lambda} Q_\lambda^* = \mathrm{ch}'\, \varphi^\lambda = \sum_{\mu \in OP_n} \frac{1}{z_\mu} 2^{\ell(\mu)/2} \varphi^\lambda(\sigma^\mu)\, p_\mu. \tag{7.1}$$

We remark that there are only two shifted tableaux with content $\mu = (n)$; both are of shape (n). Therefore, (6.6) implies $q_n = 2P_{(n)} = Q_{(n)}$. It is now easy to check via (5.5) that φ^λ agrees with the basic spin character (previously denoted by φ^n) when $\lambda = (n)$.

Theorem 7.1 (Schur [S, p. 235]). *The class functions φ^λ ($\lambda \in DP_n^+$) and $\varphi_\pm^\lambda$ ($\lambda \in DP_n^-$) are the irreducible spin characters of $\tilde{S}_n$.*

Proof. As in Schur's original proof, we will establish that (1) the φ^λ's form an orthonormal basis of Z_n', and (2) the φ^λ's are $\mathbf{Z}$-linear combinations of spin characters (i.e., virtual characters). These properties characterize the irreducible spin characters, aside from factors of ± 1. However, Corollary 6.2 implies

$$\varphi^\lambda(1) = \frac{1}{\varepsilon_\lambda} 2^{n/2}[Q_\lambda^*, p_1^n] = \frac{1}{\varepsilon_\lambda} 2^{-(n+\ell(\lambda))/2}[Q_\lambda, q_1^n]$$

$$= \frac{1}{\varepsilon_\lambda} 2^{-(n+\ell(\lambda))/2} K'_{\lambda,(1^n)} > 0, \tag{7.2}$$

and therefore φ^λ, not $-\varphi^\lambda$, is a character.

We remark that it is not hard to see from the definition that $\varphi^\lambda \downarrow \tilde{A}_n$ is **Q**-valued. Since the φ^λ's are characters, it follows that $\varphi^\lambda \downarrow \tilde{A}_n$ must be **Z**-valued.

To prove (1), observe that if $\lambda \neq \mu$, or if $\lambda = \mu \in DP_n^+$, then φ^λ and φ^μ are orthogonal on negative classes. Proposition 5.1 and (7.1) therefore imply

$$\langle \varphi^\lambda, \varphi^\mu \rangle_{\tilde{S}_n} = \tfrac{1}{2}\langle \varphi^\lambda, \varphi^\mu \rangle_{\tilde{A}_n} = [\varepsilon_\lambda^{-1} Q_\lambda^*, \varepsilon_\mu^{-1} Q_\mu^*] = \delta_{\lambda\mu}.$$

In the only remaining case, we have $\lambda = \mu \in DP_n^-$. Note that $\varphi^\lambda(\sigma) \neq 0$ only if $\sigma \in \tilde{A}_n$ or type $|\sigma| = \lambda$. Since $|C_\lambda^+| = |C_\lambda^-| = n!/z_\lambda$ and $|\varphi^\lambda(\sigma^\lambda)|^2 = z_\lambda/2$, it follows that

$$\langle \varphi_+^\lambda, \varphi_\pm^\lambda \rangle_{\tilde{S}_n} = \frac{1}{2}\langle \varphi_+^\lambda, \varphi_\pm^\lambda \rangle_{\tilde{A}_n} \pm \tfrac{1}{2} = \frac{1}{\varepsilon_\lambda^2}[Q_\lambda^*, Q_\lambda^*] \pm \frac{1}{2} = \frac{1}{2} \pm \frac{1}{2}.$$

Hence, the φ^λ's are indeed orthonormal. They are a basis of Z_n' since Lemma 4.4 shows that they span a subspace of the correct dimension.

To prove that the φ^λ's are virtual characters, we will compute the expansion of the induced characters $\theta^\mu \uparrow \tilde{S}_n$ in terms of the orthonormal φ^λ's. The analogue of this result for ordinary S_n-characters is known as Young's rule [JK].

THEOREM 7.2. *Let μ be a partition of n. We have*

$$\theta^\mu \uparrow \tilde{S}_n = \begin{cases} \varphi^\mu & \text{if } \mu \in DP_n^+ \\ \varphi_\pm^\mu & \text{if } \mu \in DP_n^- \\ 0 & \text{if } \mu \notin DP_n \end{cases} + \sum_{\lambda > \mu} c_{\lambda\mu} \begin{cases} \varphi^\lambda & \text{if } \lambda \in DP_n^+ \\ \varphi_+^\lambda + \varphi_-^\lambda & \text{if } \lambda \in DP_n^-, \end{cases}$$

where $c_{\lambda\mu} = \varepsilon_\lambda^{-1}\varepsilon_\mu^{-1} 2^{-(\ell(\lambda)+\ell(\mu))/2} K'_{\lambda\mu}$.

Proof. For any $\lambda \in DP_n$, we have

$$\begin{aligned} \tfrac{1}{2}\langle \varphi^\lambda, \theta^\mu \uparrow \tilde{S}_n \rangle_{\tilde{A}_n} &= [\mathrm{ch}'(\varphi^\lambda), \mathrm{ch}'(\theta^\mu \uparrow \tilde{S}_n)] && \text{(Prop. 5.1)} \\ &= \varepsilon_\lambda^{-1}\varepsilon_\mu^{-1}[Q_\lambda^*, 2^{-\ell(\mu)/2} q_\mu] && \text{(Prop. 5.6)} \\ &= \varepsilon_\lambda^{-1}\varepsilon_\mu^{-1} 2^{-(\ell(\lambda)+\ell(\mu))/2} K'_{\lambda\mu}. && \text{(Cor 6.2).} \end{aligned}$$

Therefore, in case λ or μ is even (so that either φ^λ or $\theta^\mu \uparrow \tilde{S}_n$ vanishes outside of $\tilde{A}_n$), we have

$$\langle \varphi^\lambda, \theta^\mu \uparrow \tilde{S}_n \rangle_{\tilde{S}_n} = \tfrac{1}{2}\langle \varphi^\lambda, \theta^\mu \uparrow \tilde{S}_n \rangle_{\tilde{A}_n} = c_{\lambda\mu}.$$

In particular, Lemma 6.3(b) implies $\langle \varphi^\mu, \theta^\mu \uparrow \tilde{S}_n \rangle = c_{\mu\mu} = 1$ for $\mu \in DP_n^+$.

In case λ and μ are both odd, we have $\langle \varphi^\lambda_\pm, \theta^\mu_+ \uparrow \tilde{S}_n \rangle = c_{\lambda\mu} \pm b_{\lambda\mu}$, where

$$b_{\lambda\mu} = \tfrac{1}{2}\langle \varphi^\lambda_+ - \varphi^\lambda_-, \theta^\mu \uparrow \tilde{S}_n \rangle = \tfrac{1}{2}\langle (\varphi^\lambda_+ - \varphi^\lambda_-) \downarrow \tilde{S}_\mu, \theta^\mu \rangle.$$

Furthermore, $b_{\lambda\mu} = 0$ unless there exists $\sigma \notin \tilde{A}_n$ with $(\varphi^\lambda_+ - \varphi^\lambda_-)(\sigma) \neq 0$ and $\theta^\mu_+(\sigma) \neq 0$. The condition $(\varphi^\lambda_+ - \varphi^\lambda_-)(\sigma) \neq 0$ forces type$|\sigma| = \lambda$, whereas Proposition 4.2(b) and Lemma 4.5 show that $\theta^\mu_+(\sigma) \neq 0$ forces type $|\sigma| = \mu$. Therefore, in the only remaining case, we have $\lambda = \mu \in DP_n^-$. Since the $\tilde{S}_\mu$-conjugacy class of σ^μ is of size $|S_\mu|/z_\mu$, and (4.4) yields $\theta^\mu_\pm(\sigma^\mu) = \varphi^\mu_\pm(\sigma^\mu)$, it follows that $b_{\mu\mu} = \frac{1}{2}$. Therefore, the multiplicity of $\varphi^\mu_\pm$ in $\theta^\mu_+ \uparrow \tilde{S}_n$ is $c_{\mu\mu} \pm b_{\mu\mu} = \frac{1}{2} \pm \frac{1}{2}$.

Finally, note that Lemma 6.3(a) justifies the restriction $\lambda \geqslant \mu$. ∎

To complete the proof of Theorem 7.1, observe that any shifted tableau of content μ has at least $\ell(\mu)$ free entries, so (6.4) implies $2^{\ell(\mu)} \mid K'_{\lambda\mu}$. Since $\lambda \geqslant \mu$ implies $\ell(\mu) \geqslant \ell(\lambda)$,

$$c_{\lambda\mu} = \frac{1}{\varepsilon_\lambda \varepsilon_\mu} 2^{(\ell(\mu) - \ell(\lambda))/2} \cdot 2^{-\ell(\mu)} K'_{\lambda\mu}$$

could fail to be an integer only if $\ell(\mu) = \ell(\lambda)$ and λ, μ are both odd. In that case, we have $c_{\lambda\mu} = 2^{-\ell(\lambda)} K'_{\lambda\mu}/2$.

If a shifted tableau T of shape λ has only $\ell(\lambda)$ free entries, then these entries consist of the main diagonal of T and nothing else. It is easy to verify (and we leave it as an exercise) that this can happen only if the content of T is a permutation of λ; in all other cases, $\mathrm{fr}(T) > \ell(\lambda)$. Hence, $c_{\lambda\mu} \in \mathbf{Z}$ unless $\lambda = \mu \in DP_n^-$.

Theorem 7.2 therefore describes an integer, triangular transition matrix between $\theta^\mu \uparrow \tilde{S}_n$ ($\mu \in DP_n$) and φ^λ that has a unit diagonal. This matrix may be inverted over $\mathbf{Z}$, thereby proving that the φ^λ's are virtual characters. ∎

A shifted tableau is said to be *standard* if it has no marked letters and uses each unmarked letter 1, 2, ..., n exactly once. Let $g^\lambda = 2^{-n} K'_{\lambda,(1^n)}$ denote the number of standard shifted tableaux of shape λ. From (7.2), we have

COROLLARY 7.3. $\deg \varphi^\lambda = \varepsilon_\lambda^{-1} 2^{(n-\ell(\lambda))/2} g^\lambda$.

There is an explicit formula for g^λ due to Schur [S, p. 235]:

$$g^\lambda = \frac{n!}{\lambda_1! \cdots \lambda_l!} \prod_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

Macdonald's proof of this identity [Ma, p. 135] uses the definition of g^λ given above; it does not use properties of Q-functions.

In order to determine the irreducible spin characters of $\tilde{A}_n$, we first compute the difference character Δ^λ of the self-associate character φ^λ ($\lambda \in DP_n^+$).

THEOREM 7.4 (Schur [S, p. 236]). *Let $\lambda \in DP_n^+$. We have*

$$\Delta^\lambda(\sigma) = \pm i^{(n-\ell(\lambda))/2}\sqrt{z_\lambda} \qquad \textit{if } \text{type}\,|\sigma| = \lambda,$$

and in all other cases, $\Delta^\lambda(\sigma) = 0$.

Proof. Choose k sufficiently large so that $\mu = (2k, \lambda_1, \lambda_2, ...) \in DP^-$, and form the reduced Clifford product $\theta = \varphi_\pm^{2k} \times_c \varphi^\lambda$. Since one of the factors is self-associate, Theorem 5.3 and (7.1) imply

$$\text{ch}'(\theta \uparrow \tilde{S}_{n+2k}) = \frac{1}{\sqrt{2}}\, Q^*_{(2k)} Q^*_\lambda.$$

By Proposition 5.1 and the fact that μ is odd, we have

$$\langle \theta \uparrow \tilde{S}_{n+2k}, \varphi_+^\mu + \varphi_-^\mu \rangle = [Q^*_{(2k)} Q^*_\lambda, Q^*_\mu] = [P_{(2k)} P_\lambda, Q_\mu].$$

By Lemma 6.3, the "leading term" of both P_μ and $P_{(2k)}P_\lambda$ is the monomial x^μ. Since Q_μ is dual to P_μ, we conclude that $[P_{(2k)}P_\lambda, Q_\mu] = 1$. Since the φ^μ's are irreducible, it follows that $\langle \theta \uparrow \tilde{S}_{n+2k}, \varphi_+^\mu \rangle = 1$ and $\langle \theta \uparrow \tilde{S}_{n+2k}, \varphi_-^\mu \rangle = 0$, or vice versa. In particular

$$\langle \varphi_\pm^{2k} \times_c \varphi^\lambda, (\varphi_+^\mu - \varphi_-^\mu) \downarrow \tilde{S}_{(2k,n)} \rangle = \pm 1. \tag{7.3}$$

Note that $(\varphi_+^\mu - \varphi_-^\mu)(\sigma) \neq 0$ only if type $|\sigma| = \mu$. Assuming $2k > n$, the only $\tilde{S}_{(2k,n)}$-classes that include elements of type μ are the split classes $C^\pm_{((2k),\lambda)}$. Proposition 4.2(b) and (7.3) therefore imply

$$\frac{2}{|\tilde{S}_{(2k,n)}|}\, |C^+_{((2k),\lambda)}| \cdot \varphi_\pm^{2k}(\sigma^{(2k)})\, \Delta^\lambda(\sigma^\lambda) \cdot 2\bar{\varphi}_\pm^\mu(\sigma^\mu) = \pm 1.$$

Solving for $\Delta^\lambda(\sigma^\lambda)$ yields the claimed formula.

To complete the proof, we must show that $\Delta^\lambda(\sigma) = 0$ if type $|\sigma| \neq \lambda$. To see this, note that the classes of type λ contribute

$$\frac{2}{|\tilde{A}_n|} \cdot \frac{n!}{z_\lambda} \cdot |\Delta^\lambda(\sigma^\lambda)|^2 = 2$$

to $\|\Delta^\lambda\|^2_{\tilde{A}_n}$. But we know $\|\Delta^\lambda\|^2_{\tilde{A}_n} = 2$ by Lemma 4.1, so there can be no other classes for which Δ^λ is nonzero. ∎

By Frobenius reciprocity, each irreducible $\tilde{A}_n$-module is a constituent of some irreducible $\tilde{S}_n$-module. Lemma 4.1 therefore implies

COROLLARY 7.5. *The irreducible spin characters of* $\tilde{A}_n$ *are* $\varphi^\lambda \downarrow \tilde{A}_n$ $(\lambda \in DP_n^-)$ *and* $\frac{1}{2}(\varphi^\lambda \pm \Delta^\lambda) \downarrow \tilde{A}_n$ $(\lambda \in DP_n^+)$.

Of course, to write down the character table of $\tilde{A}_n$, one needs to know how Δ^λ behaves on the even classes of $\tilde{S}_n$ that are split by $\tilde{A}_n$. According to Theorem 2.7, these are the classes C_μ and $C_\mu^\pm$ indexed by $\mu \in DP_n^+$. Since $\sigma \mapsto \sigma_1 \sigma \sigma_1^{-1}$ is an automorphism of $\tilde{A}_n$ that permutes the split classes, it follows that $\Delta^\lambda(\sigma) = -\Delta^\lambda(\sigma_1 \sigma \sigma_1^{-1})$. This gives us sufficient information to determine the character table, since Theorem 7.4 shows that the split classes of $\tilde{A}_n$ for which $\Delta^\lambda \neq 0$ are of a *unique* type.

8. A SHIFTED ANALOGUE OF THE LITTLEWOOD-RICHARDSON RULE

Since the P_λ's form a **Z**-basis of Ω (Corollary 6.2), we may define integers $f_{\mu\nu}^\lambda$ for each λ, μ, $\nu \in DP$ via

$$P_\mu P_\nu = \sum_\nu f_{\mu\nu}^\lambda P_\lambda .$$

The theory we have developed implies that, aside from powers of 2, the integers $f_{\mu\nu}^\lambda$ count character multiplicities related to induced Clifford products and are therefore nonnegative. More precisely, we claim:

THEOREM 8.1. *Let* $\mu \in DP_k$, $\nu \in DP_{n-k}$, $\lambda \in DP_n$, *and form the reduced Clifford product* $\varphi^\mu \times_c \varphi^\nu$. *We have*

$$\langle (\varphi^\mu \times_c \varphi^\nu) \uparrow \tilde{S}_n, \varphi^\lambda \rangle = \frac{1}{\varepsilon_\lambda \varepsilon_{\mu \cup \nu}} 2^{(\ell(\mu) + \ell(\nu) - \ell(\lambda))/2} f_{\mu\nu}^\lambda,$$

unless λ *is odd and* $\lambda = \mu \cup \nu$ *(multiset partition union). In that case, the multiplicity of* $\varphi_\pm^\lambda$ *is* 0 *or* 1 *according to choice of associates.*

Proof. By Theorem 5.3 and (7.1), we have $\mathrm{ch}'((\varphi^\mu \times_c \varphi^\nu) \uparrow \tilde{S}_n) = \varepsilon_{\mu \cup \nu}^{-1} Q_\mu {}^* Q_\nu^*$. Therefore, Proposition 5.1 implies

$$\begin{aligned} \tfrac{1}{2}\langle (\varphi^\mu \times_c \varphi^\nu) \uparrow \tilde{S}_n, \varphi^\lambda \rangle_{\tilde{A}_n} &= \varepsilon_{\mu \cup \nu}^{-1} \varepsilon_\lambda^{-1} [Q_\mu^* Q_\nu^*, Q_\lambda^*] \\ &= \varepsilon_{\mu \cup \nu}^{-1} \varepsilon_\lambda^{-1} 2^{(\ell(\mu) + \ell(\nu) - \ell(\lambda))/2} f_{\mu\nu}^\lambda. \qquad (8.1) \end{aligned}$$

Furthermore, we have

$$\langle (\varphi^\mu \times_c \varphi^\nu) \uparrow \tilde{S}_n, \varphi^\lambda \rangle_{\tilde{S}_n} = \tfrac{1}{2}\langle (\varphi^\mu \times_c \varphi^\nu) \uparrow \tilde{S}_n, \varphi^\lambda \rangle_{\tilde{A}_n},$$

unless there exists $\sigma \notin \tilde{A}_n$ such that $(\varphi^\mu \times_c \varphi^\nu)(\sigma) \neq 0$ and $\varphi^\lambda(\sigma) \neq 0$. The condition $\varphi^\lambda(\sigma) \neq 0$ forces λ to be odd and type $|\sigma| = \lambda$, whereas Proposition 4.2(b) and Theorem 7.4 imply that the condition $(\varphi^\mu \times_c \varphi^\nu)(\sigma) \neq 0$ forces type $|\sigma| = \mu \cup \nu$. In that case (i.e., $\lambda = \mu \cup \nu$, λ odd), $\varphi^\lambda_+ + \varphi^\lambda_-$ vanishes on odd classes, so (8.1) implies

$$\langle (\varphi^\mu \times_c \varphi^\nu) \uparrow \tilde{S}_n, \varphi^\lambda_+ + \varphi^\lambda_- \rangle_{\tilde{S}_n} = f^\lambda_{\mu\nu}.$$

But the leading term of both $P_\mu P_\nu$ and P_λ is the monomial x^λ(cf. Cor. 6.2), so $f^\lambda_{\mu\nu} = 1$ in this case. Since the φ^λ's are irreducible, the multiplicities of φ^λ_+ and φ^λ_- must be 0 and 1 (or vice versa). ▌

The coefficients $c^\lambda_{\mu\nu}$ that appear in the analogous Schur function expansion

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$$

describe the decomposition of outer tensor products induced from $S_k \times S_{n-k}$ to S_n. These coefficients have an explicit combinatorial interpretation known as the Littlewood–Richardson (or simply LR) rule [Ma, JK].

In this section, we will use the theory of shifted tableaux developed by Sagan and Worley to derive an analogous rule for the $f^\lambda_{\mu\nu}$'s. However, before we can describe this shifted LR rule, we first require the introduction of shifted versions of skew diagrams and tableaux.

A *shifted skew diagram* is a collection of boxes of the form $D'_{\lambda/\mu} := D'_\lambda - D'_\mu$, where D'_μ and D'_λ are any pair of shifted diagrams with $D'_\mu \subseteq D'_\lambda$. A *shifted skew tableau* T of *shape* λ/μ is an assignment $T: D'_{\lambda/\mu} \to \mathbf{P}'$ satisfying the usual shifted rules in (6.2).

Extend the definitions of Q_λ's and P_λ's to skew diagrams via

$$Q_{\lambda/\mu}(x) = \sum_{T: D'_{\lambda/\mu} \to \mathbf{P}'} x^T, \qquad P_{\lambda/\mu}(x) = \sum_{T: D'_{\lambda/\mu} \to \mathbf{P}'}{}' x^T$$

summed over shifted (skew) tableaux T, where the restricted sum $\sum'$ includes only those T with unmarked main diagonals. Note that $Q_{\lambda/\mu} = 2^{\ell(\lambda) - \ell(\mu)} P_{\lambda/\mu}$. In case $D'_\mu \not\subseteq D'_\lambda$, it is convenient to formally define $Q_{\lambda/\mu} = P_{\lambda/\mu} = 0$.

The practical reader will complain that it is unclear from these definitions that we have $Q_{\lambda/\mu}, P_{\lambda/\mu} \in \Omega$. Nor is it clear that these generating functions are symmetric. However, these objections can be dismissed by means of the identity

$$Q_\lambda(x, y) = \sum_{\mu \in DP} Q_\mu(x) Q_{\lambda/\mu}(y), \tag{8.2}$$

$$
\begin{array}{cccc} & & 1' & 1 \\ & 1 & 1 & 2' \\ 2 & 2 & & \\ & 3 & & \end{array}
\qquad
\begin{array}{cccc} & & 1 & 1 \\ & 1 & 2 & 2 \\ 1 & 2 & & \\ & 3 & & \end{array}
\qquad
\begin{array}{cccc} & & 1 & 1 \\ & 1 & 2' & 2 \\ 1 & 2 & & \\ & 3 & & \end{array}
\qquad
\begin{array}{cccc} & & 1 & 1 \\ & 1' & 2' & 2 \\ 1 & 2 & & \\ & 3 & & \end{array}
$$

FIGURE 2

which follows directly from our definition of Q_λ. Note that $Q_\lambda(x, y) \in \Omega(x) \otimes \Omega(y)$ is used as an abbreviation for $Q_\lambda(x_1, x_2, ..., y_1, y_2, ...)$. One may deduce $Q_{\lambda/\mu} \in \Omega$ from this identity since $Q_{\lambda/\mu}(y)$ is the coefficient of $Q_\mu(x)$ in $Q_\lambda(x, y)$. We claim that as a further consequence of this identity, one may deduce that the coefficients $f^\lambda_{\mu\nu}$ also appear in the expansion

$$Q_{\lambda/\mu} = \sum_{\nu \in DP} f^\lambda_{\mu\nu} Q_\nu. \tag{8.3}$$

That is, we claim:

PROPOSITION 8.2 (Macdonald [Ma, III. 5]). $[P_\mu P_\nu, Q_\lambda] = [P_\nu, Q_{\lambda/\mu}]$.

Proof. Introduce a third set of indeterminates $z = (z_1, z_2, ...)$ and consider

$$\begin{aligned}
\sum_{\lambda, \mu \in DP} Q_\mu(x)\, Q_{\lambda/\mu}(y)\, P_\lambda(z) &= \sum_{\lambda \in DP} Q_\lambda(x, y)\, P_\lambda(z) \\
&= \prod_{i,j} \frac{1 + x_i z_j}{1 - x_i z_j} \cdot \frac{1 + y_i z_j}{1 - y_i z_j} \\
&= \Big(\sum_{\mu \in DP} Q_\mu(x)\, P_\mu(z) \Big) \Big(\sum_{\nu \in DP} Q_\nu(y)\, P_\nu(z) \Big),
\end{aligned}$$

by successive applications of (8.2) and (6.5). Extract the coefficient of $Q_\mu(x)\, Q_\nu(y)\, P_\lambda(z)$. ∎

As a consequence, observe that $f^\lambda_{\mu\nu} = 0$ unless $D'_\mu, D'_\nu \subseteq D'_\lambda$.

Since $f^\lambda_{\mu\nu} \geqslant 0$, (8.3) predicts that there is a content-preseving algorithm $S \to T$ whose input consists of the shifted tableaux S of shape λ/μ and whose output consists of shifted tableaux T of various shapes ν. Furthermore, (8.3) also implies that there is an algorithm of this type that "factors uniformly"; i.e., the number of inputs S that produce a given output T may be assumed to depend only on λ/μ and ν, not on the choice of T. Given such an algorithm, one may choose a fixed output tableau T_ν for each shape ν, and use the linear independence of the Q_ν's to deduce that $f^\lambda_{\mu\nu}$ is the number of shifted tableaux $S\colon D'_{\lambda/\mu} \to \mathbf{P}'$ such that $S \to T_\nu$.

Sagan and Worley have constructed an explicit algorithm $S \rightarrow^J T$ that meets these requirements; it is a shifted analogue of Schützenberger's *jeu de taquin*. We remark in particular that a proof of the crucial "uniform factorization" property for this algorithm can be found in [Wo, p. 116]. We will prove that it is indeed possible to explicitly describe the tableaux S for which $S \rightarrow^J T_\nu$ for a certain (well-chosen) target T_ν, and thus obtain a shifted LR rule.

Define the *word* $w(S) = w_1 w_2 \cdots$ of a (possibly skew) shifted tableau S to be the sequence obtained by reading the rows of S from left to right, starting with the last row. For example, the word of the first tableau in Fig. 2 is $322112'1'1$. Given any word $w = w_1 w_2 \cdots w_n$ over the alphabet $\mathbf{P}'$, define a series of statistics $m_i(j)(0 \leqslant j \leqslant 2n,\ i \geqslant 1)$ depending on w as follows:

$$m_i(j) = \text{multiplicity of } i \text{ among } w_{n-j+1}, \ldots, w_n \qquad (0 \leqslant j \leqslant n)$$

$$m_i(n+j) = m_i(n) + \text{multiplicity of } i' \text{ among } w_1, \ldots, w_j \qquad (0 < j \leqslant n). \tag{8.4}$$

In particular, $(m_1(2n), m_2(2n), \ldots)$ is the content of w and $m_i(0) = 0$. The multiplicities m_i may be computed in real time by scanning the word twice—first from right to left, then from left to right. During the right-to-left phase, m_i monitors the accumulation of i, and during the left-to-right phase, m_i monitors the accumulation of i'. Note that m_i is not reset between phases.

The word w is said to satisfy the *lattice property* if, whenever $m_i(j) = m_{i-1}(j)$, we have

$$w_{n-j} \neq i, i' \qquad \text{if } 0 \leqslant j < n \tag{8.5a}$$

$$w_{j-n+1} \neq i-1, i' \qquad \text{if } n \leqslant j < 2n. \tag{8.5b}$$

Note that w_{n-j} or w_{j-n+1} is the next letter to be read after the jth step. For example, the words of the tableaux in Fig. 2 all satisfy the lattice property. We remark that by induction, the lattice property implies $m_1(j) \geqslant m_2(j) \geqslant \cdots$ for $0 \leqslant j \leqslant 2n$, and so it is reminiscent of the lattice property that appears in the ordinary LR rule.

Finally, we are ready to state the shifted LR rule. (In property (2) below, $|w|$ denotes the word obtained by erasing the marks of w.)

THEOREM 8.3. *The coefficient $f^\lambda_{\mu\nu}$ is the number of shifted tableaux S of shape λ/μ and content ν such that*

(1) *$w = w(S)$ satisfies the lattice property;*

(2) *the leftmost i of $|w|$ is unmarked in w $(1 \leqslant i \leqslant \ell(\nu))$.*

We note that the entries i referred to in (2) are always free in S. Furthermore, it is possible to prove that the lattice property is unaffected when the markings of these extremal entries are changed arbitrarily. Therefore, we may drop condition (2) with the understanding that the coefficient so described would instead be $2^{\ell(\nu)}f^{\lambda}_{\mu\nu} = [Q_{\lambda/\mu}, Q_{\nu}]$.

For an example, take $\lambda = (6, 5, 2, 1)$, $\mu = (4, 2)$, $\nu = (4, 3, 1)$. One finds $f^{\lambda}_{\mu\nu} = 4$, and the corresponding tableaux are those appearing in Fig. 2.

Our first step will consist of a review of only those aspects of the Sagan–Worley theory that will be needed in our proof. We make no attempt to give a complete survey. In particular, the reader may find it disconcerting that we will never need to explicitly describe their tableau algorithm $S \to^{J} T$.

One may recover S from knowledge only of $w(S)$ and the shape of S, but more significantly, Sagan and Worley have proved that the algorithm $S \to^{J} T$ depends not on the shape of S, but only on $w(S)$. Hence, it may be regarded as a word-algorithm $w \to^{J} T(w)$, and in this form coincides with their shifted analogue of the Robinson–Schensted correspondence [Sa, p. 96; Wo, p. 94]. In particular, we note that if w is the word of a (nonskew) shifted tableau T, then $T = T(w)$.

Define an equivalence relation $\sim$ on $\mathbf{P}'$-words w by setting

$$w \sim w' \qquad \text{iff } T(w) = T(w');$$

i.e., $w \sim w'$ iff w and w' produce the same tableau. Sagan and Worley have shown that a small set of generators for $\sim$ may be explicitly described. These generators are particularly simple if we restrict our attention to those w for which $T(w)$ is standard. In that case, w must be a permutation of $1, 2, ..., n$ (no marked letters), and the restriction $\sim|_{S_n}$ may be described [Sa, p. 74; Wo, p. 78] as the transitive closure of

$$\cdots bac \cdots \sim \cdots bca \cdots \qquad (a < b < c) \tag{8.6a}$$

$$\cdots acb \cdots \sim \cdots cab \cdots \qquad (a < b < c) \tag{8.6b}$$

$$ac \cdots \sim ca \cdots \qquad (a < c). \tag{8.6c}$$

That is, two adjacent letters (a, c) of w may be interchanged if they appear at the beginning of w, or if there is an adjacent intermediate witness (b). The relations (8.6a) and (8.6b) first appeared in the work of Knuth [K] in connection with the Robinson–Schensted correspondence.

We summarize this discussion with the following:

LEMMA 8.4. *Choose a standard shifted tableau T_ν of shape ν. Then $f^{\lambda}_{\mu\nu}$ is the number of standard shifted tableaux S of shape λ/μ such that $w(S) \sim w(T_\nu)$.*

1	2	3	4	5	6	7		1	1	1	1	1	1	1
	8	9	10	11					2	2	2	2		
		12	13							3	3			
		(a)								(b)				

FIGURE 3

Define a permutation $w = w_1 \cdots w_n$ of distinct letters to be a *hook* or *hook-shaped* if the letters $\{w_j: w_j > w_1\}$ (resp., $\{w_j: w_j < w_1\}$) form an increasing (resp., decreasing) subsequence of w. The element w_1 is defined to be the *extreme point* of the hook w.

For any permutation w, let $h(w)$ denote the size of the largest hook-shaped subsequence of w. For example, $h(6742531) = 5$, since 67431 is a largest subhook. Also, given any $A \subseteq P$, let $w|_A$ denote the subsequence of w formed by the letters $w_j \in A$.

LEMMA 8.5. *If I is an interval of consecutive letters, then $h(w|_I)$ is $\sim$-invariant; i.e., $w \sim w'$ implies $h(w|_I) = h(w'|_I)$.*

Proof. Since I is an interval, (8.6) shows that $w \sim w'$ implies $w|_I \sim w'|_I$. Therefore, it is sufficient to prove $h(w) = h(w')$, or even $h(w) \leqslant h(w')$.

Suppose $w|_A$ is a hook of size $h(w)$, and assume $w \sim w'$ (or $w' \sim w$) is one of the relations in (8.6). We note that $w'|_A$ is also a hook unless w and w' are related by (8.6) and (8.6b) and we have a, $c \in A$, $b \notin A$. In these cases, we let $A' = A \cup \{b\} - \{a\}$ when bac or cab is a subsequence of w, and we let $A' = A \cup \{b\} - \{c\}$ when bca or acb is a subsequence of w. In either case, it is clear that $w'|_{A'}$ is a hook, and therefore $h(w) \leqslant h(w')$. ∎

For each $v \in DP_n$, we choose T_v to be the standard shifted tableau whose entries are numbered consecutively by rows from 1 to n, starting with the first row. For example, the tableau T_v corresponding to $v = (7, 4, 2)$ appears in Fig. 3a. Henceforth, we fix v and let R_j denote the interval of letters assigned to the jth row by T_v.

LEMMA 8.6. $w \sim w(T_v)$ *if and only if*

$$h(w|_{R_j}) = h(w|_{R_j \cup R_{j+1}}) = v_j \qquad (j \geqslant 1). \tag{8.7}$$

Proof. Let T be a nonskew standard shifted tableau. If $T \neq T_v$, we claim that $w = w(T)$ cannot satisfy (8.7). To see this, suppose that $i \in R_j$ is the smallest entry that occurs in different rows of T and T_v.

If i appears above row j of T, then $(w|_{R_{j-1}})i$ is an increasing subhook of $w|_{R_{j-1} \cup R_j}$ of length $v_{j-1}+1$, violating (8.7).

If i appears below row j of T, then i cannot be among the two smallest elements of R_j, since these must be assigned to the boxes (j, j) and $(j, j+1)$ by both T and T_v. Therefore, $w|_{R_j}$ cannot be a hook, since i occurs to the left of an increasing pair of smaller elements. Hence, $h(w|_{R_j}) < |R_j| = v_j$, which violates (8.7) and proves our claim.

Conversely, it is easy to check that $w(T_v)$ does satisfy (8.7), so we have proved the lemma in the case that $w = w(T)$ is a tableau-word. However, Lemma 8.5 shows that (8.7) is $\sim$-invariant, and we know (from the definition) that tableau-words are a collection of equivalence class representatives for $\sim$. ▌

Let $w = w_1 \cdots w_n$ be any permutation such that $w|_{R_j}$ is a hook for $j \geqslant 1$, and let e_j denote the extreme point of $w|_{R_j}$. Define a labeling operation $w \mapsto L(w)$ that produces a $\mathbf{P}'$-word of content v by setting $L(w) = \hat{w} = \hat{w}_1 \cdots \hat{w}_n$, where

$$\hat{w}_k = \begin{cases} j & \text{if } w_k \in R_j,\ w_k \geqslant e_j \\ j' & \text{if } w_k \in R_j,\ w_k < e_j. \end{cases}$$

For example, if $v = (4, 3, 1)$ and $w = 82615734$, then $L(w) = 3121'2'211$.

LEMMA 8.7. *Let w be a permutation such that $w|_{R_j}$ is a hook. Then $w \sim w(T_v)$ iff $L(w)$ satisfies the lattice property.*

Proof. Assume $w \sim w(T_v)$ and let m_i $(i \geqslant 1)$ denote the statistics (8.4) for $\hat{w} = L(w)$. For any fixed j $(0 \leqslant j \leqslant 2n)$, let $M_i = M_i(j) \subseteq R_i$ denote the first $m_i(j)$ letters of $w|_{R_i}$ that are read during the scanning process associated with (8.4). To prove $\hat{w}$ satisfies the lattice property, suppose $m_{i-1}(j) = m_i(j)$ and assume that one of the conditions (8.5) is violated. We claim that there is a subhook $w|_A$ of $w|_{R_{i-1} \cup R_i}$ that contradicts Lemma 8.6. To see this, consider $A = R_{i-1} \cup \{a\} \cup M_i - M_{i-1}$, where a is the next letter of w to be read during the scanning process. We have $\hat{a} = i$ or i' if $0 \leqslant j < n$, and $\hat{a} = i-1$ or i' if $n \leqslant j < 2n$. In either case, one may check that $w|_A$ is a hook of size $v_{i-1}+1$ that violates (8.7), thus proving our claim.

For the converse, assume that $\hat{w}$ satisfies the lattice property. By Lemma 8.6, it is enough to prove $h(w|_{R_{i-1} \cup R_i}) = v_{i-1}$ for $i > 1$. Therefore, let $w|_A$ be any maximal subhook of $w|_{R_{i-1} \cup R_i}$. Without loss of generality, we may assume that the extreme point of $w|_A$ is the extreme point of either $w|_{R_{i-1}}$ or $w|_{R_i}$.

In case the extreme point of $w|_A$ is that of $w|_{R_{i-1}}$, let w_j be the largest letter in $R_{i-1} \cap A$. Since A is maximal, we have $R_{i-1} \cap A = \{w_k \in R_{i-1} : w_k \leqslant w_j\}$ and

$$|R_{i-1} \cap A| = v_{i-1} - m_{i-1}(n-j).$$

The letters of $R_i \cap A$ must form an increasing subsequence of $w_{j+1}, ..., w_n$. Since the label i' is attached to letters of w that appear in decreasing order, maximality of A also implies

$$|R_i \cap A| = m_i(n-j) + (0 \text{ or } 1),$$

with the alternative 0 or 1 depending on whether some (unique) $w_k \in R_i \cap A$ satisfies $\hat{w}_k = i'$. Since the lattice property implies $m_{i-1} \geqslant m_i$, we may deduce $|A| \leqslant v_{i-1}$ unless it is possible to satisfy both $|R_i \cap A| = m_i(n-j)+1$ and $m_i(n-j) = m_{i-1}(n-j)$ simultaneously. However, (8.5a) shows that it is not.

In case the extreme point of $w|_A$ is that of $w|_{R_i}$, let w_j be the smallest letter in $R_i \cap A$. Since A is maximal, we have $R_i \cap A = \{w_k \in R_i : w_k \geqslant w_j\}$, and

$$|R_i \cap A| = m_i(n+j).$$

The letters of $R_{i-1} \cap A$ must form a decreasing subsequence of $w_{j+1}, ..., w_n$. Since the label $i-1$ is attached to letters of w that appear in increasing order, maximality of A also implies

$$|R_{i-1} \cap A| = v_{i-1} - m_{i-1}(n+j) + (0 \text{ or } 1),$$

with the alternative 0 or 1 depending on whether some (unique) $w_k \in R_{i-1} \cap A$ satisfies $\hat{w}_k = i-1$. Therefore, we may conclude $|A| \leqslant v_{i-1}$, since (8.5b) guarantees that it is impossible to satisfy both $|R_{i-1} \cap A| = v_{i-1} - m_{i-1}(n+j)+1$ and $m_{i-1}(n+j) = m_i(n+j)$ simultaneously. Thus, we have verified that the lattice property implies (8.7), so the proof is complete. ∎

Proof of Theorem 8.3. Lemmas 8.4 and 8.7 show that $f^{\lambda}_{\mu\nu}$ is the number of standard shifted tableaux S of shape λ/μ such that $w(S)|_{R_j}$ is a hook (of size v_j) and $L(w(S))$ satisfies the lattice property. Given such an S, let $\hat{S}: D'_{\lambda/\mu} \to \mathbf{P}'$ denote the assignment of content v induced by the labeling $w \mapsto L(w)$. There is at most one i per column of $\hat{S}$, since they appear in increasing order in $w(S)$; similarly, there is at most one i' per row. Furthermore, the extreme points of $w|_{R_j}$ are not marked in $\hat{S}$. Thus, we conclude that $\hat{S}$ is a shifted tableau of content v such that (1) $w(\hat{S})$ satisfies the lattice property, and (2) the extreme points of $w(S)$ are not marked. These properties clearly characterize the tableaux that arise via the operation $S \mapsto \hat{S}$. ∎

We remark that one may deduce from this proof and [Sa1, Thm. 9.3] that the tableaux S that appear in Theorem 8.3 are precisely the tableaux for which $S \to^J \hat{T}_v$, where $\hat{T}_v$ is of the form appearing in Fig. 3b.

9. Inner Tensor Products

The (pointwise) product of characters gives the space $Z_n \oplus Z'_n$ of $\tilde{S}_n$-class functions the structure of a $\mathbf{Z}_2$-graded commutative ring. Consider the subring formed by $Z_n \oplus Z_n^+$, where Z_n^+ denotes the subspace of Z'_n spanned by $\{1'_\lambda : \lambda \in OP_n\}$. Since $\mathrm{ch}' : Z_n^+ \to \Omega^n_{\mathbf{C}}$ is a linear isomorphism, we may thus give the external direct sum $\Lambda^n_{\mathbf{C}} \oplus \Omega^n_{\mathbf{C}}$ a $\mathbf{Z}_2$-graded ring structure by insisting that

$$Z_n \oplus Z_n^+ \xrightarrow{\mathrm{ch} \oplus \mathrm{ch}'} \Lambda^n_{\mathbf{C}} \oplus \Omega^n_{\mathbf{C}}$$

defines a (graded) ring isomorphism.

Let $*$ denote the resulting commutative, associative product on $\Lambda^n_{\mathbf{C}} \oplus \Omega^n_{\mathbf{C}}$; it generalizes Littlewood's so-called "inner product" on Λ^n [L2]. To describe $*$ more directly, use the identifications $f = (f, 0) \in \Lambda^n_{\mathbf{C}} \oplus \Omega^n_{\mathbf{C}}$ for $f \in \Lambda^n_{\mathbf{C}}$ and $g' = (0, g) \in \Lambda^n_{\mathbf{C}} \oplus \Omega^n_{\mathbf{C}}$ for $g \in \Omega^n_{\mathbf{C}}$. From the products

$$1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda; \qquad 1'_\lambda 1_\mu = \delta_{\lambda\mu} 1'_\lambda \ (\lambda \in OP_n); \qquad 1'_\lambda 1'_\mu = \delta_{\lambda\mu} 1_\lambda \ (\lambda, \mu \in OP_n)$$

and the definitions of the characteristic maps (Section 5), we see that

$$\begin{aligned} p_\lambda * p_\mu &= z_\lambda \delta_{\lambda\mu} p_\lambda \\ p'_\lambda * p_\mu &= z_\lambda \delta_{\lambda\mu} p'_\lambda && (\lambda \in OP_n) \\ p'_\lambda * p'_\mu &= 2^{-\ell(\lambda)} z_\lambda \delta_{\lambda\mu} p_\lambda && (\lambda, \mu \in OP_n) \end{aligned}$$

completely determines the ring structure of $\Lambda^n_{\mathbf{C}} \oplus \Omega^n_{\mathbf{C}}$.

We remark that $\mathrm{ch} \oplus \mathrm{ch}'$ is also a linear map of $Z_n \oplus Z'_n$ onto $\Lambda^n_{\mathbf{C}} \oplus \Omega^n_{\mathbf{C}}$, but in this form it is not a ring homomorphism. (Consider the characteristic of $1'_\lambda \cdot 1'_\lambda = 1_\lambda$ for $\lambda \in DP_n^-$). In spite of this, observe that since $\ker(\mathrm{ch}') \cdot Z_n \subseteq \ker(\mathrm{ch}')$, we may still assert that

$$\mathrm{ch}'(\varphi\chi) = \mathrm{ch}' \varphi * \mathrm{ch}\, \chi \tag{9.1}$$

for arbitrary $\varphi \in Z'_n$, $\chi \in Z_n$.

If $f, g \in \Lambda^n_{\mathbf{C}} \oplus \Omega^n_{\mathbf{C}}$ are the characteristics of $\tilde{S}_n$-characters, then $f * g$ encodes the most essential information need to count irreducible multiplicities in the inner tensor (or Kronecker) product of the corresponding $\tilde{S}_n$-representations. Specifically, this information is carried by the s_λ-expansion of $f * g$ (if $f, g \in \Lambda^n_{\mathbf{C}}$) or $f' * g'$ (if $f, g \in \Omega^n_{\mathbf{C}}$), and by the Q^*_λ-expansion of $f * g'$ (if $f \in \Lambda^n_{\mathbf{C}}$, $g \in \Omega^n_{\mathbf{C}}$). However, even for ordinary $\mathbf{C}S_n$-modules, there has only been limited success in computing such decompositions (see, e.g. [GR]).

In the following we will use the shifted LR rule to derive an explicit description of the multiplicities that occur in the tensor product of the basic spin representation of $\tilde{S}_n$ with any irreducible $\mathbf{C}S_n$-module. By the above discussion, this is equivalent to the Q_μ^*-expansion of $q'_n * s_\lambda$.

PROPOSITION 9.1. *If* $f \in \Lambda^n$, $g \in \Omega^n$, *then* $[q'_n * f, g'] = \langle f, g \rangle$.

Proof. Note that (5.4) implies $q'_n * p_\lambda = 2^{\ell(\lambda)} p'_\lambda (\lambda \in OP_n)$ and $q'_n * p_\lambda = 0$ $(\lambda \notin OP_n)$. Hence, $[q'_n * p_\lambda, p'_\mu] = z_\lambda \delta_{\lambda\mu} = \langle p_\lambda, p_\mu \rangle$ for $\mu \in OP_n$ and arbitrary λ. Appeal to linearity. ∎

Consider the family of symmetric functions $S_\lambda \in \Omega^n$ $(|\lambda| = n)$ defined by $S'_\lambda = q'_n * s_\lambda$. These symmetric functions may be characterized in several different ways. For example,

PROPOSITION 9.2. (a) $\prod_{i,j} ((1 + x_i y_j)/(1 - x_i y_j)) = \sum_\lambda s_\lambda(x) S_\lambda(y)$.

(b) $S_\lambda(x) = \sum_{T: D_\lambda \to \mathbf{P}'} x^T$, *summed over all tableaux* T *of unshifted shape* λ *satisfying the shifted rules in* (6.2).

Proof. The coefficient of s_λ in p_μ is $\langle s_\lambda, p_\mu \rangle$, by the orthonormality of Schur functions. It follows that for $\mu \in OP_n$,

$$p_\mu(x) = \sum_{|\lambda| = n} [S_\lambda, p_\mu]\, s_\lambda(x),$$

since Proposition 9.1 implies $\langle s_\lambda, p_\mu \rangle = [S_\lambda, p_\mu]$. Therefore,

$$\sum_{\mu \in OP_n} \frac{1}{z_\mu} 2^{l(\mu)} p_\mu(x)\, p_\mu(y) = \sum_{|\lambda| = n} s_\lambda(x)\, S_\lambda(y).$$

Apply Proposition 5.2 to obtain (a).

The identity in (b) follows, for example, from Worley's bijective proof of (a), taking (b) as a definition [Wo, (5.1)]. ∎

We remark that another characterization of S_λ is the determinant

$$S_\lambda = \det[q_{\lambda_i - i + j}],$$

where $q_{-r} = 0$ for $r > 0$. This definition is shown to be equivalent to Proposition 9.2(a) in [Ma, III.4] via the Jacobi–Trudi identity. It is shown to be equivalent to Proposition 9.2(b) in [Wo, (2.11)]. We further remark that the equivalence of our definition $(S'_\lambda = q'_n * s_\lambda)$ with these various characterizations is hidden implicitly in [Mo4; R].

Since the s_μ's are a $\mathbf{Z}$-basis of Λ, we may define integers $g_{\lambda\mu}$ $(\lambda \in DP_n$, $|\mu| = n)$ via the expansion

$$P_\lambda = \sum_{|\mu| = n} g_{\lambda\mu} s_\mu.$$

By Proposition 9.1, we have $\langle s_\mu, P_\lambda \rangle = [S_\mu, P_\lambda]$, so these coefficients also appear in the expansion

$$S_\mu = \sum_{\lambda \in DP_n} g_{\lambda\mu} Q_\lambda.$$

Since $S'_\mu = q'_n * s_\mu$, these coefficients count, aside from powers of 2, the multiplicities of spin characters φ^λ in the product $\varphi^n \chi^\mu$ ($\chi^\mu = \mu$th irreducible S_n-character), and are therefore nonnegative. This representation-theoretic interpretation of $g_{\lambda\mu}$ has been known to Morris and Stanley [Sa2], but apparently has not been previously published. Furthermore, Sagan [Sa1, p. 78] and Worley [Wo, p. 121] have shown that their combinatorial theory provides another explanation of the fact that $g_{\lambda\mu} \geqslant 0$, but without a shifted LR rule, an interpretation of $g_{\lambda\mu}$ could not be made completely explicit.

THEOREM 9.3. *Let $\lambda \in DP_n$, $|\mu| = n$.*

(a) *We have*

$$\langle \varphi^n \chi^\mu, \varphi^\lambda \rangle = \frac{1}{\varepsilon_\lambda \varepsilon_{(n)}} 2^{(\ell(\lambda)-1)/2} g_{\lambda\mu},$$

unless $\lambda = (n)$, n is even, and μ is a hook-partition. In that case, the multiplicity of $\varphi^\lambda_\pm$ is 0 or 1 according to choice of associates.

(b) *$g_{\lambda\mu}$ is the number of "shifted tableaux" S of unshifted shape μ and content λ such that*

(1) *$w = w(S)$ satisfies the lattice property.*

(2) *The leftmost i of $|w|$ is unmarked in w $(1 \leqslant i \leqslant \ell(\lambda))$.*

For example, one finds $P_{(3,1)} = s_{(3,1)} + s_{(2,2)} + s_{(2,1,1)}$, and the relevant tableaux are

$$\begin{matrix} 1 & 1 & 1 \\ 2 & & \end{matrix}\;, \qquad \begin{matrix} 1' & 1 \\ 1 & 2 \end{matrix}\;, \qquad \begin{matrix} 1' & 1 \\ 1 & \\ 2 & \end{matrix}$$

We remark that, as in the shifted LR rule, one may drop condition (2) with the understanding that the coefficient so described would be $2^{\ell(\lambda)} g_{\lambda\mu} = \langle s_\mu, Q_\lambda \rangle = [S_\mu, Q_\lambda]$.

Proof. For (b), observe that Proposition 9.2(b) imlies that $S_\lambda = Q_{\lambda+\delta/\delta}$, where $\delta = (l, l-1, ..., 2, 1)$ and $l = \ell(\lambda)$. Apply the shifted LR rule in Theorem 8.3.

For (a), note that (9.1) implies

$$\mathrm{ch}'(\varphi^n\chi^\mu) = \mathrm{ch}'\, \varphi^n * \mathrm{ch}\, \chi^\mu = \frac{1}{\varepsilon_{(n)}\sqrt{2}}\, q'_n * s_\mu = \frac{1}{\varepsilon_{(n)}\sqrt{2}}\, S'_\mu.$$

By Proposition 5.1 and (7.1), it follows that

$$\frac{1}{2}\langle \varphi^n\chi^\mu, \varphi^\lambda\rangle_{\tilde{A}_n} = \frac{1}{\varepsilon_\lambda\varepsilon_{(n)}\sqrt{2}}\,[S_\mu, Q^*_\lambda] = \frac{1}{\varepsilon_\lambda\varepsilon_{(n)}}\, 2^{(\ell(\lambda)-1)/2} g_{\lambda\mu}. \tag{9.2}$$

Furthermore, we have

$$\langle \varphi^n\chi^\mu, \varphi^\lambda\rangle_{\tilde{S}_n} = \tfrac{1}{2}\langle \varphi^n\chi^\mu, \varphi^\lambda\rangle_{\tilde{A}_n}$$

unless there exists $\sigma \notin \tilde{A}_n$ such that $\varphi^n(\sigma)\,\chi^\mu(|\sigma|)\,\varphi^\lambda(\sigma) \neq 0$.

The conditions $\varphi^n(\sigma) \neq 0$ and $\varphi^\lambda(\sigma) \neq 0$ force n to be even and $\mathrm{type}|\sigma| = (n) = \lambda$. By the Murnaghan–Nakayama rule [JK], $\chi^\mu(w) \neq 0$ and $\mathrm{type}(w) = (n)$ forces μ to be hook-shaped; i.e., $\mu = (k, 1, 1, ...)$ for some $k \geqslant 1$. In case these constraints (n even, $\lambda = (n)$, $\mu =$ hook) are met, then $\varphi^\lambda_+ + \varphi^\lambda_-$ vanishes on odd classes, (9.2) implies $\langle \varphi^n\chi^\mu, \varphi^\lambda_+ + \varphi^\lambda_-\rangle = g_{\lambda\mu}$, and (b) implies $g_{\lambda\mu} = 1$. Since the φ^λ's are irreducible, the multiplicities of φ^λ_+ and φ^λ_- must be 0 and 1 (or vice versa). ∎

As a final remark, we mention that for any $f \in \Omega^n$, (5.4) implies $q'_n * f' = f$. (Take $f = p_\mu$.) In particular, $q'_n * P'_\lambda = P_\lambda$, so the coefficients $g_{\lambda\mu}$ also describe the S_n-character expansion of $\varphi^n\varphi^\lambda$.

References

[A] G. E. Andrews, "The Theory of Partitions," Addison-Wesley, Reading, MA, 1976.

[CR] C. W. Curtis and I. Reiner, "Methods of Representation Theory," Vol. I, Wiley, New York, 1981.

[GR] A. M. Garsia and J. Remmel, Shuffles of permutations and the Kronecker product, *Graphs Combin.* **1** (1985), 217–263.

[H] J. F. Humphreys, Blocks of projective representations of symmetric groups, *J. London Math. Soc.* **33** (1986), 441–452.

[JK] G. D. James and A. Kerber, "The Representation Theory of the Symmetric Group," Addison-Wesley, Reading, MA, 1981.

[K] D. E. Knuth, Permutations, matrices and generalized Young tableaux, *Pacific J. Math.* **34** (1970), 709–727.

[L1] D. E. Littlewood, On certain symmetric functions, *Proc. London Math. Soc. (3)* **11** (1961), 485–498.

[L2] D. E. Littlewood, The Kronecker product of symmetric group representations, *J. London Math. Soc.* **31** (1956), 89–93.

[Ma] I. G. Macdonald, "Symmetric Functions and Hall Polynomials," Oxford Univ. Press, Oxford, 1979.

[Mo1] A. O. MORRIS, The spin representation of the symmetric group, *Proc. London Math. Soc. (3)* **12** (1962), 55–76.

[Mo2] A. O. MORRIS, The spin representation of the symmetric group, *Canad. J. Math.* **17** (1965), 543–549.

[Mo3] A. O. MORRIS, A survey on Hall–Littlewood functions and their applications to representation theory, *in* "Combinatoire et représentation du groupe symétrique" (D. Foata, Ed.), Lecture Notes in Math., Vol. 579, Springer-Verlag, Berlin, 1977.

[Mo4] A. O MORRIS, The projective characters of the symmetric group—an alternative proof, *J. London Math. Soc. (2)* **19** (1979), 57–58.

[N] M. L. NAZAROV, An orthogonal basis of irreducible projective representations of the symmetric group, *Functional Anal. Appl.* **22** (1988), 66–68.

[R] E. W. READ, On the projective characters of the symmetric group, *J. London Math. Soc. (2)* **15** (1977), 456–464.

[Sa1] B. E. SAGAN, Shifted tableaux, Schur Q-functions and a conjecture of R. Stanley, *J. Combin. Theory Ser. A* **45** (1987), 62–103.

[Sa2] B. E. SAGAN, private communication.

[S] I. SCHUR, Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen, *J. Reine Angew. Math.* **139** (1911), 155–250.

[St] R. P. STANLEY, Theory and application of plane partitions I, II, *Stud. Appl. Math.* **50** (1971), 167–188, 259–279.

[W] H. WEYL, "The Classical Groups," Princeton Univ. Press, Princeton, NJ, 1946.

[Wo] D. R. WORLEY, "A Theory of Shifted Young Tableaux," Ph. D. thesis, MIT, 1984.